Topological entropy of maps on inverse limits

Ana Anušić
University of São Paulo, Brazil

Coauthors: Chris Mouron (Rhodes College, Memphis, TN, USA)

Virtual VCDS
June 14th 2021

Joint work

A., C. Mouron, *Strongly commuting interval maps*, preprint 2020, arXiv:2010.15328 [math.DS]

A., C. Mouron, *Topological entropy of diagonal maps on inverse limit spaces*, preprint 2020, arXiv:2010.15332 [math.DS]

Inverse limits

We are assuming:

X_i are compact, connected, metric spaces (continua) for $i \geq 0$,

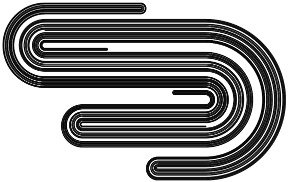
$f_i : X_i \to X_{i-1}$ are continuous and onto, for $i \geq 0$.

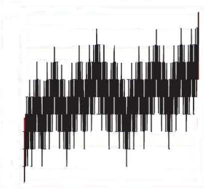
The inverse limit space is given by

$$\varprojlim\{X_i, f_i\} := \{(x_0, x_1, x_2, \dots) : f_i(x_i) = x_{i-1}, i \geq 1\} \subset \prod_{i=0}^{\infty} X_i,$$

equipped with the product topology.

Denote by $\pi_i : \varprojlim\{X_i, f_i\} \to X_i$ the coordinate projections. They are continuous.





Motivating question

Question

Compute the topological entropy of a map $\Psi\colon \varprojlim\{X_i, f_i\} \to \varprojlim\{X_i, f_i\}$.

For example, assume that X_i are all equal to some continuum X, and f_i are all equal to some map $f\colon X \to X$. Assume that Ψ is the **natural extension** $\hat{f}\colon \varprojlim\{X, f\} \to \varprojlim\{X, f\}$,

$$\hat{f}((x_0, x_1, x_2 \dots)) = (f(x_0), f(x_1), f(x_2), \dots) = (f(x_0), x_0, x_1, \dots).$$

$$\begin{array}{ccccccccc}
X & \xleftarrow{f} & X & \xleftarrow{f} & X & \xleftarrow{f} & X & \xleftarrow{f} & \cdots \\
\downarrow f & & \downarrow f & & \downarrow f & & \downarrow f & & \\
X & \xleftarrow[f]{} & X & \xleftarrow[f]{} & X & \xleftarrow[f]{} & X & \xleftarrow[f]{} & \cdots
\end{array}$$

Then $\mathrm{Ent}(\hat{f}) = \mathrm{Ent}(f)$ (Bowen 1970).

Straight-down maps on $\varprojlim\{X_i, f_i\}$

Assume there exists a commutative diagram

$$\begin{array}{ccccccccc} X_0 & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & X_2 & \xleftarrow{f_3} & X_3 & \xleftarrow{f_4} & \cdots \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \\ X_0 & \xleftarrow[f_1]{} & X_1 & \xleftarrow[f_2]{} & X_2 & \xleftarrow[f_3]{} & X_3 & \xleftarrow[f_4]{} & \cdots \end{array}$$

Let $\Psi\colon \varprojlim\{I, f_i\} \to \varprojlim\{I, f_i\}$ be the **straight-down map**:

$$\Psi((x_0, x_1, x_2, \dots)) := (g_0(x_0), g_1(x_1), g_2(x_2), \dots)$$

Then $\mathrm{Ent}(\Psi) = \sup_i \mathrm{Ent}(g_i)$ (Ye 1995).

Theorem (Mouron 2012)

Every straight-down map on the pseudo-arc has entropy 0 or ∞.

Given $r \in [0, \infty]$, there exists a pseudo-arc homeomorphism h such that $\mathrm{Ent}(h) = r$. (Boroński, Činč, Oprocha 2021)

Other continuous self-maps of $\varprojlim\{X_i, f_i\}$

Not all continuous maps on $\varprojlim\{X_i, f_i\}$ are straight-down maps. For example, the diagram

$$\begin{array}{ccccccccc} X_0 & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & X_2 & \xleftarrow{f_3} & X_3 & \xleftarrow{f_4} & \cdots \\ \downarrow g_0 & & \downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 & & \\ X_0 & \xleftarrow[f_1]{} & X_1 & \xleftarrow[f_2]{} & X_2 & \xleftarrow[f_3]{} & X_3 & \xleftarrow[f_4]{} & \cdots \end{array}$$

can only ε_i-commute, where $\varepsilon_i \to 0$ as $i \to \infty$.

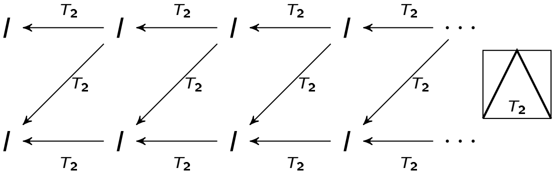
Or, it can be of the form (here $f_n^m = f_{n+1} \circ \ldots f_m \colon X_m \to X_n$):

$$\begin{array}{ccccccccc} X_{n_0} & \xleftarrow{f_{n_0}^{n_1}} & X_{n_1} & \xleftarrow{f_{n_1}^{n_2}} & X_{n_2} & \xleftarrow{f_{n_2}^{n_3}} & X_{n_3} & \xleftarrow{f_{n_3}^{n_4}} & \cdots \\ \swarrow g_0 & & \swarrow g_1 & & \swarrow g_2 & & \swarrow g_3 & & \\ X_{m_0} & \xleftarrow[f_{m_0}^{m_1}]{} & X_{m_1} & \xleftarrow[f_{m_1}^{m_2}]{} & X_{m_2} & \xleftarrow[f_{m_2}^{m_3}]{} & X_{m_3} & \xleftarrow[f_{m_3}^{m_4}]{} & \cdots \end{array}$$

Or, the diagram can only ε_i-commute (Mioduszewski 1963).

Ye's result?

In general, Ye's result does not hold when the map is not straight-down:



$$(x_0, x_1, x_2, \ldots) \mapsto (T_2(x_1), T_2(x_2), T_2(x_3), \ldots) = (x_0, x_1, x_2, \ldots),$$

so $\mathrm{Ent}(g) = \mathrm{Ent}(id) = 0$. However, $\mathrm{Ent}(T_2) = \log(2)$.

Straight-down components

Let $\Psi\colon \varprojlim\{X_i, f_i\} \to \varprojlim\{X_i, f_i\}$ be continuous.

For $i \geq 0$ define $\psi_i\colon X_i \to 2^{X_i}$ as $\psi_i(x) = \pi_i \circ \Psi \circ \pi_i^{-1}(x)$.

Set-valued!

$$
\begin{array}{ccccccccc}
X_0 & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & X_2 & \xleftarrow{f_3} & X_3 & \xleftarrow{f_4} & \cdots \\
\downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \\
X_0 & \xleftarrow[f_1]{} & X_1 & \xleftarrow[f_2]{} & X_2 & \xleftarrow[f_3]{} & X_3 & \xleftarrow[f_4]{} & \cdots
\end{array}
$$

The diagram "commutes" in a sense that $f_i^j \circ \psi_j(x) \subset \psi_i \circ f_i^j(x)$ for every $i < j$ and $x \in X_j$.

Theorem (A., Mouron 2020)

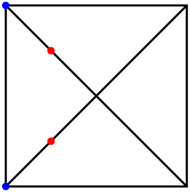
$\mathrm{Ent}(\Psi) \leq \liminf_{i \geq 0} \mathrm{Ent}(\psi_i)$ *(entropy of set-valued maps?)*

The equality does not hold in general (nor the limit exists). However, there is a wide class for which the equality holds.

Dynamics of set-valued maps

Let $F\colon X \to 2^X$ be a (set-valued) function such that the **graph** $\Gamma(F) = \{(x,y) : y \in F(x)\}$ is closed in $X \times X$.

For $n \in \mathbb{N}$, an n-orbit is $(x_1, \dots, x_n)$ such that $x_{i+1} \in F(x_i)$ for every $1 \le i < n$. Denote by $Orb_n(F)$ the set of all n-orbits of F.



$F(0) = \{0, 1\}$, $F(1/4) = \{1/4, 3/4\}$

$(0, 1, 1, 0, 1, 1)$ is a 6-orbit

$(1/4, 1/4, 3/4, 3/4, 1/4)$ is a 5-orbit

$F = T_2^{-1} \circ T_2$

Set-valued entropy (Alvin, Kelly, Raines, Tennant, ...)

For $n \in \mathbb{N}$ and $\varepsilon > 0$, we say that a set $S \subset Orb_n(F)$ is (n, ε)-separated if for every $(x_1, \dots, x_n), (y_1, \dots, y_n) \in S$ there exists $i \in \{1, \dots, n\}$ such that $d_X(x_i, y_i) > \varepsilon$. Let $s_{n,\varepsilon}(F)$ denote the largest cardinality of an (n, ε)-separated set.

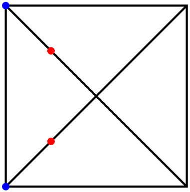
For $n \in \mathbb{N}$ and $\varepsilon > 0$, we say that a set $S \subset Orb_n(F)$ is (n, ε)-spanning if for every $(x_1, \dots, x_n) \in Orb_n(F)$ there exists $(y_1, \dots, y_n) \in S$ such that $d_X(x_i, y_i) < \varepsilon$ for every $i \in \{1, \dots, n\}$. The smallest cardinality of an (n, ε)-spanning set is denoted by $r_{n,\varepsilon}(F)$.

The **entropy** of F is defined as

$$\mathrm{Ent}(F) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(s_{n,\varepsilon}(F)) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(r_{n,\varepsilon}(F)).$$

Example

$$\mathrm{Ent}(F) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log(s_{n,\varepsilon}(F)) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log(r_{n,\varepsilon}(F)).$$



$F(0) = \{0,1\}$, $F(1/4) = \{1/4, 3/4\}$

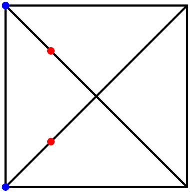
$(0,1,1,0,1,1)$ is a 6-orbit

$(1/4, 1/4, 3/4, 3/4, 1/4)$ is a 5-orbit

$\{0,1\}^n \subset Orb_n(F)$ is (n,ε)-separated for $\varepsilon < 1$, so $\mathrm{Ent}(F) \geq \log(2)$.
$\{\{x, 1-x\}^n : x \in \{0, \frac{\varepsilon}{2}, \varepsilon, \frac{3\varepsilon}{2}, \ldots, \lfloor\frac{1}{\varepsilon}\rfloor\frac{\varepsilon}{2}\}\} \subset Orb_n(F)$ is (n,ε)-spanning, so $\mathrm{Ent}(F) \leq \log(2)$.

Thus $\mathrm{Ent}(F) = \log(2)$.

Example



$F(0) = \{0, 1\}$, $F(1/4) = \{1/4, 3/4\}$

$(0, 1, 1, 0, 1, 1)$ is a 6-orbit

$(1/4, 1/4, 3/4, 3/4, 1/4)$ is a 5-orbit

$\mathrm{Ent}(F) = \log(2)$

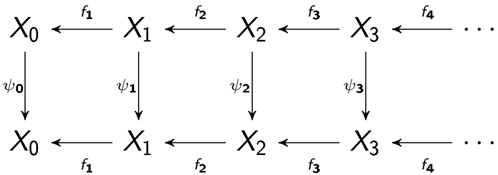
Note that $\mathrm{Ent}(F^k) = \mathrm{Ent}(F) = \log(2)$ for every $k \in \mathbb{N}$. Thus for $k \geq 2$, $\mathrm{Ent}(F^k) \neq k\mathrm{Ent}(F)$.

Proposition (A., Mouron 2020)

$$\mathrm{Ent}(F) \geq \sup_{n \in \mathbb{N}} \frac{\mathrm{Ent}(F^n)}{n}.$$

Recall...

Let $\Psi\colon \varprojlim\{X_i, f_i\} \to \varprojlim\{X_i, f_i\}$ be continuous.
For $i \geq 0$ define $\psi_i\colon X_i \to X_i$ as $\psi_i(x) = \pi_i \circ \Psi \circ \pi_i^{-1}(x)$.
Set-valued!

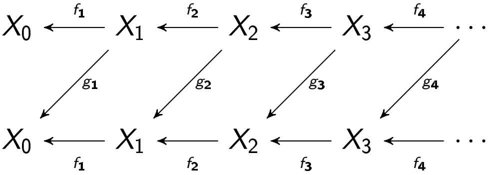


Theorem (A., Mouron 2020)

$\mathrm{Ent}(\Psi) \leq \liminf_{i\geq 0} \mathrm{Ent}(\psi_i)$.

Diagonal maps

Assume that the following diagram commutes:



Define $G\colon \varprojlim\{I, f_i\} \to \varprojlim\{I, f_i\}$ by

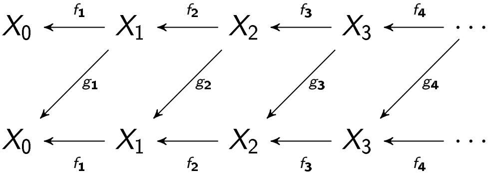
$$G((x_0, x_1, x_2, \dots)) = (g_1(x_1), g_2(x_2), g_3(x_3), \dots).$$

Straight-down components: $\psi_i(x) = \pi_i \circ G \circ \pi_i^{-1}(x) = g_{i+1} \circ f_{i+1}^{-1}(x)$ for $i \geq 0$. It follows that $\mathrm{Ent}(G) \leq \liminf_{i\to\infty} \mathrm{Ent}(g_i \circ f_i^{-1})$.

Question

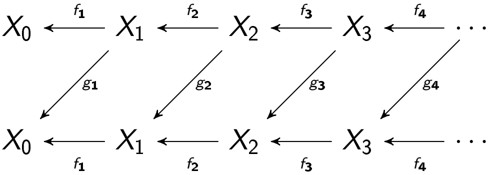
Given continuous maps $f, g\colon X \to X$, how to compute $Ent(g \circ f^{-1})$?

Diagonal maps



(a) used to construct an example of a tree-like continuum without a fixed point property (for which there is a self-map without a fixed point) Oversteegen and Rogers 1980, Hoehn and Hernández-Gutiérrez 2018,

(b) used by Mouron in 2018 to give an example of an exact map (also called locally eventually onto) on the pseudo-arc.



Proposition (A., Mouron 2020)

Assume that maps $f_i, g_i\colon X_{i+1} \to X_i$, $i \in \mathbb{N}$, *satisfy:*

(i) $g_i \circ f_{i+1} = f_i \circ g_{i+1}$, *and*

(ii) $g_{i+1} \circ f_{i+1}^{-1} = f_i^{-1} \circ g_i$, $i \in \mathbb{N}$.

Let $X = \varprojlim(X_i, f_i)$ *and* $G\colon X \to X$ *be the diagonal map*

$$G((x_0, x_1, x_2, \dots)) = (g_1(x_1), g_2(x_2), g_3(x_3), \dots).$$

Then $\mathrm{Ent}(g_i \circ f_i^{-1}) \le \mathrm{Ent}(G)$ *for every* $i \in \mathbb{N}$. *In particular, it follows that*

$$\mathrm{Ent}(G) = \lim_{i\to\infty} \mathrm{Ent}(g_i \circ f_i^{-1}).$$

Condition $g_{i+1} \circ f_{i+1}^{-1} = f_i^{-1} \circ g_i$?

Sketch of proof:
For every n-orbit $\mathbf{x} = (x_1, \dots, x_n)$ of $g_i \circ f_i^{-1}$ we can find $\xi \in \varprojlim\{X_i, f_i\}$ such that $\pi_i(G^k(\xi)) = x_{k+1}$, $1 \le k < n$.

$$\begin{array}{ccccccccc}
x_1 & \xleftarrow{f_i} & y_1^1 & \xleftarrow{f_{i+1}} & y_1^2 & \xleftarrow{f_{i+2}} & y_1^3 & \xleftarrow{f_{i+3}} & \cdots \\
& \swarrow{\scriptstyle g_i} & & \swarrow{\scriptstyle g_{i+1}} & & \swarrow{\scriptstyle g_{i+2}} & & & \\
x_2 & \xleftarrow{f_i} & y_2^1 & \xleftarrow{f_{i+1}} & y_2^2 & \xleftarrow{f_{i+2}} & \cdots & & \\
& \swarrow{\scriptstyle g_i} & & \swarrow{\scriptstyle g_{i+1}} & & & & & \\
x_3 & \xleftarrow{f_i} & y_3^1 & \xleftarrow{f_{i+1}} & \cdots & & & & \\
& \swarrow{\scriptstyle g_i} & & & & & & & \\
x_4 & \xleftarrow{f_i} & \cdots & & & & & &
\end{array}$$

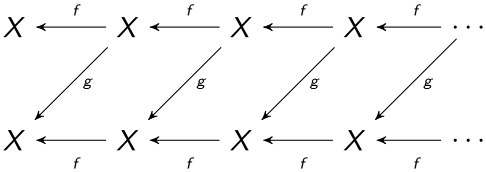
Now for an (n, ε)-separated set $\{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset Orb_n(g_i \circ f_i^{-1})$ we find corresponding $\xi_1, \dots, \xi_n \in \varprojlim\{X_i, f_i\}$. Then the set $\{(G^k(\xi_1))_{0 \le k < n}, \dots, (G^k(\xi_n))_{0 \le k < n}\} \subset Orb_n(G)$ is $(n, \varepsilon/2^i)$-separated.

Condition $g_{i+1} \circ f_{i+1}^{-1} = f_i^{-1} \circ g_i$?

Note that if $g_i \circ f_{i+1} = f_i \circ g_{i+1}$, then $g_{i+1} \circ f_{i+1}^{-1} \subseteq f_i^{-1} \circ g_i$.

How difficult is it to satisfy the equality?

Restrict to sequences of the same map first.



Assume that $f \circ g = g \circ f$, $g \circ f^{-1} = f^{-1} \circ g$, and
$G\colon \varprojlim\{X, f\} \to \varprojlim\{X, f\}$ is given by $G((x_i)_{i\geq 0}) = (g(x_i))_{i\geq 1}$. Then
$\mathrm{Ent}(G) = \mathrm{Ent}(g \circ f^{-1})$.

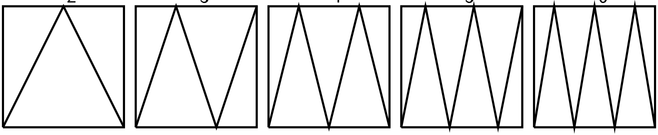
Note that if $f \circ g = g \circ f$, then $g \circ f^{-1} \subseteq f^{-1} \circ g$. Which commuting maps satisfy the equality?

Strongly commuting maps

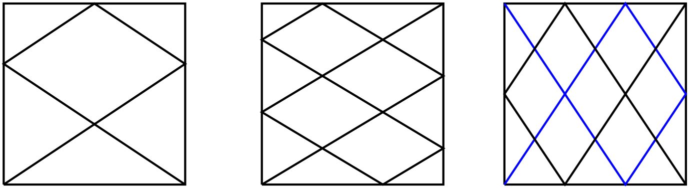
Maps $f, g \colon X \to X$ are called **strongly commuting** if $g \circ f^{-1} = f^{-1} \circ g$.

For $n \geq 2$, let $T_n \colon I \to I$ be the **symmetric *n*-tent map**:

T_2 T_3 T_4 T_5 T_6



$T_3^{-1} \circ T_2 = T_2 \circ T_3^{-1}$ $T_3 \circ T_5^{-1} = T_5^{-1} \circ T_3$ $T_4^{-1} \circ T_6 \neq T_6 \circ T_4^{-1}$

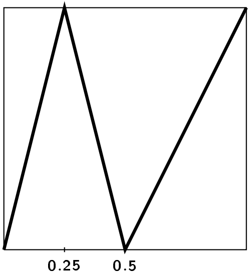


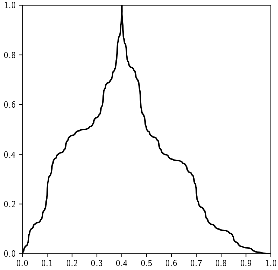
Proposition (A., Mouron 2020)

T_n and T_m strongly commute if and only if n and m are relatively prime.

Entropy and strongly commuting open maps

Let $F_n, F_m \colon I \to I$ be piecewise (strictly) monotone onto maps with n and m pieces of monotonicity. Assume that F_n and F_m are additionally **open**.





Proposition (A., Mouron 2020)

If F_n and F_m strongly commute, then
$\mathrm{Ent}(F_n \circ F_m^{-1}) = \max\{log(n), \log(m)\} = \max\{\mathrm{Ent} F_n, \mathrm{Ent} F_m\}$.

If n, m are relatively prime, then $\mathrm{Ent}(T_n \circ T_m^{-1}) = \max\{\log(n), \log(m)\}$.

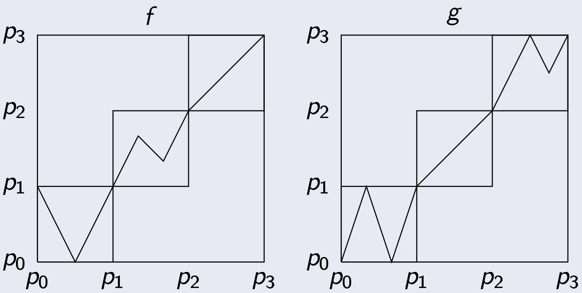
Strongly commuting maps (on the interval)

Theorem (A., Mouron 2020)

Let $f, g\colon I \to I$ be **piecewise monotone** *onto maps which strongly commute. Then there are $0 = p_0 < p_1 < \ldots < p_k = 1$ such that $[p_i, p_{i+1}]$ is invariant under f^2 and g^2 for every $i \in \{0, 1, \ldots, k-1\}$, and such that one of the following occurs:*

(i) *$f^2|_{[p_i,p_{i+1}]}$ and $g^2|_{[p_i,p_{i+1}]}$ are both open and non-monotone,*

(ii) *$f^2|_{[p_i,p_{i+1}]}$ is monotone, or* **(iii)** *$g^2|_{[p_i,p_{i+1}]}$ is monotone.*



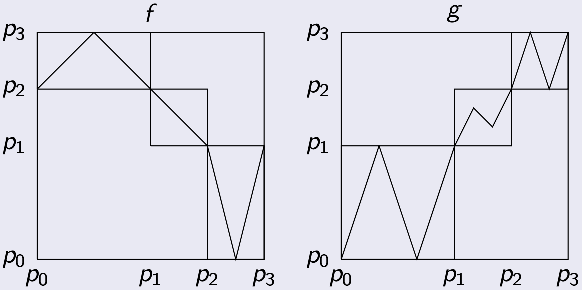
Strongly commuting maps (on the interval)

Theorem (A., Mouron 2020)

Let $f, g \colon I \to I$ be **piecewise monotone** *onto maps which strongly commute. Then there are $0 = p_0 < p_1 < \ldots < p_k = 1$ such that $[p_i, p_{i+1}]$ is invariant under f^2 and g^2 for every $i \in \{0, 1, \ldots, k-1\}$, and such that one of the following occurs:*

(i) $f^2|_{[p_i,p_{i+1}]}$ *and* $g^2|_{[p_i,p_{i+1}]}$ *are both open and non-monotone,*

(ii) $f^2|_{[p_i,p_{i+1}]}$ *is monotone, or* **(iii)** $g^2|_{[p_i,p_{i+1}]}$ *is monotone.*



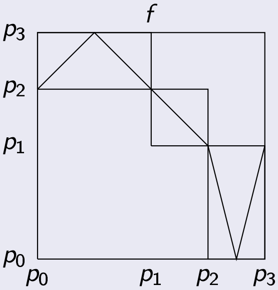
Strongly commuting maps (on the interval)

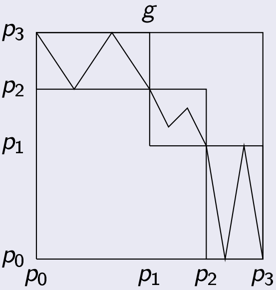
Theorem (A., Mouron 2020)

Let $f, g\colon I \to I$ be **piecewise monotone** *onto maps which strongly commute. Then there are $0 = p_0 < p_1 < \ldots < p_k = 1$ such that $[p_i, p_{i+1}]$ is invariant under f^2 and g^2 for every $i \in \{0, 1, \ldots, k-1\}$, and such that one of the following occurs:*

(i) $f^2|_{[p_i, p_{i+1}]}$ *and* $g^2|_{[p_i, p_{i+1}]}$ *are both open and non-monotone,*

(ii) $f^2|_{[p_i, p_{i+1}]}$ *is monotone, or* **(iii)** $g^2|_{[p_i, p_{i+1}]}$ *is monotone.*





Theorem (A., Mouron 2020)

If $f, g\colon I \to I$ are piecewise monotone onto maps which strongly commute, then $\mathrm{Ent}(g \circ f^{-1}) = \max\{\mathrm{Ent}(f), \mathrm{Ent}(g)\}$.

Sketch of proof:

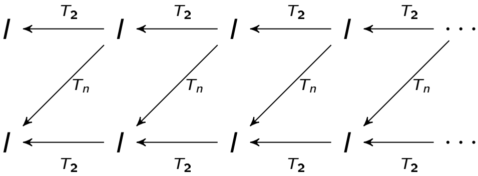
$2\mathrm{Ent}(g \circ f^{-1}) \overset{!}{=} \mathrm{Ent}((g \circ f^{-1})^2) = \mathrm{Ent}(g^2 \circ f^{-2}) = \max\{\mathrm{Ent}(g^2 \circ f^{-2}|_{[p_i, p_{i+1}]}) : i \in \{0, 1, \ldots, k-1\}\} = \max\{\mathrm{Ent}(g^2), \mathrm{Ent}(f^2)\} = 2\max\{\mathrm{Ent}(g), \mathrm{Ent}(f)\}$.

In particular, given piecewise monotone onto maps $f, g\colon I \to I$ which strongly commute, the entropy of the diagonal map $G\colon \varprojlim\{I, f\} \to \varprojlim\{I, f\}$ given by $G((x_0, x_1, x_2, \ldots)) = (g(x_1), g(x_2), \ldots)$ equals

$$\mathrm{Ent}(G) = \mathrm{Ent}(g \circ f^{-1}) = \max\{\mathrm{Ent}(f), \mathrm{Ent}(g)\}.$$

Example

Let $X_2 = \varprojlim\{I, T_2\}$ be the Knaster continuum. For every **odd** $n > 1$ we can construct a map $G_n \colon X_2 \to X_2$ with $\mathrm{Ent}(G_n) = \log(n)$:



Recall also that every homeomorphism on X_2 has entropy $k\log(2)$ for $k \in \mathbb{N}$ (Bruin and Štimac 2013 for unimodal inverse limits in general).

Thank you!