# Rotations and Birkhoff sums

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## Older results/personal history

## Non-$L^1$ functions with convergent Birkhoff averages

Example of P. Major. Answering a question of M. Laczkovich:

Z.B.: If $S, T : X \to X$ are two $\mu$-ergodic transformations which generate a free $\mathbb{Z}^2$ action on the finite non-atomic Lebesgue measure space $(X, \mathcal{S}, \mu)$ then for any $c_1, c_2 \in \mathbb{R}$ there exists a $\mu$-measurable function $f : X \to \mathbb{R}$ such that

$M_N^S f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(S^j x) \to c_1$, and $M_N^T f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(T^j x) \to c_2$,

$\mu$ almost every $x$ as $N \to \infty$.

Two different irrational rotations generate a free $\mathbb{Z}^2$ action on $\mathbb{T}$ $\Rightarrow$ answer to Laczkovich's question.

Trying two answer Laczkovich's question first I proved the following theorem:

**T** .: *Let $f:\mathbb{R}\to\mathbb{R}$ be a given measurable function, periodic by 1.*

*For an $\alpha\in\mathbb{R}$ put* $$M_n^\alpha f(x)=\frac{1}{n+1}\sum_{k=0}^{n} f(x+k\alpha).$$

*Let $\Gamma_f$ denote the set of those $\alpha$s in $(0,1)$ for which $M_n^\alpha f(x)$ converges for almost every $x\in\mathbb{R}$.*

*Then from $|\Gamma_f|>0$ it follows that $f$ is integrable on $[0,1]$.*

$|\Gamma_f|>0 \Rightarrow f\in L^1$ and for all $\alpha\in[0,1]\backslash\mathbb{Q}$ the limit of $M_n^\alpha f(x)$ equals $\int_0^1 f$ by the Birkhoff Ergodic thm.

I gave an example of $f\notin L^1$ for which $\dim_H\Gamma_f=1$, but of course $|\Gamma_f|=0$.

With G. Keszthelyi: generalizations.

$G$ is a compact Abelian topological group, $m$ is the Haar measure.

$G$ is connected if and only if $\widehat{G}$ is torsion-free.

Given a strictly monotone increasing sequence of integers $(n_k)$ we consider the non-conventional ergodic averages $$M_N^\alpha f(x)=\frac{1}{N+1}\sum_{k=0}^{N} f(x+n_k\alpha).$$

Of course, if $n_k=k$ we have the usual Birkhoff averages.

The $f$-rotation set is

$\Gamma_f=\{\alpha\in G : M_N^\alpha f(x)$ converges for $m$ a.e. $x$ as $N\to\infty\}$.

If $G=\mathbb{T}$, $m=\lambda$, the Lebesgue measure on $\mathbb{T}$, and $n_k=k$ then for any measurable $f:\mathbb{T}\to\mathbb{R}$ from $m(\Gamma_f)>0$ it follows that $f\in L^1(\mathbb{T})$.

Scrutinizing the proof of this result one can see that the set

$\Gamma_{f,0}=\left\{\alpha\in G : \frac{f(x+n_k\alpha)}{k}\to 0 \text{ for } m \text{ a.e. } x\right\}$ played an important role.

It is obvious that $\Gamma_f\subset\Gamma_{f,0}$.

We will also use the slightly larger set

$$\Gamma_{f,b}=\left\{\alpha\in G : \limsup_{k\to\infty}\frac{|f(x+n_k\alpha)|}{k}<\infty \text{ for } m \text{ a.e. } x\right\}.$$

**T** .: *If $(n_k)$ is a strictly monotone increasing sequence of integers and $G$ is a compact, locally connected Abelian group and $f:G\to\mathbb{R}$ is a measurable function then from $m(\Gamma_{f,b})>0$ it follows that $f\in L^1(G)$.*

**Rem**.: Since $\Gamma_{f,b}\supset\Gamma_{f,0}\supset\Gamma_f$ the theorem implies that if one considers the non-conventional ergodic averages $M_N^\alpha f$ on a locally compact Abelian group for group rotations and $m(\Gamma_f)>0$ then $f\in L^1(G)$.

# Maximizing points and coboundaries for an irrational rotation on the Circle

This is based on a joint paper with Julien Brémont.

Let $(X,T)$ be a top. dyn. sys., where $X$ is a compact metric space and $T : X \to X$ a continuous and surjective transformation.
Fix a continuous function $f : X \to \mathbb{R}$.

$$f_n(x) = \sum_{k=0}^{n-1} T^k f(x) = \sum_{k=0}^{n-1} f \circ T^k(x), \quad n \geq 1.$$

The optimal pointwise growth of $(f_n(x))$ is an important question arising naturally.

**D** .: Let $f : X \to \mathbb{R}$ be continuous. A point $x_0 \in X$ is *maximizing* for $f$ if there exists a constant $C \geq 0$ such that: $\star \quad \forall x \in X, \forall n \geq 1, \quad f_n(x) \leq f_n(x_0) + C.$

The point $x_0$ is *exactly maximizing* if one can take $C = 0$ and *weakly maximizing* if $C$ is replaced by $C(x)$.
If $\mu$ is a fixed Borel probability measure, we also say that $x_0$ is "$\mu$-*weakly maximizing*" if $\star$ is true for $\mu$-a.e $x$ with a constant $C(x)$.

Recall: The existence of maximizing points is naturally the first question to be adressed. If $f = c + g - Tg$ with $g$ bounded and $c$ constant then clearly every point is maximizing for $f$. It is natural to ask whether this is the only situation.

The answer is negative for dynamical systems where the
Lemma of Mané-Conze-Guivarc'h is valid.
In this case any Hölder continuous $f$ admits a maximizing point.
We denote by $C_{m0}(\mathbb{T})$ the set of those functions in $C(\mathbb{T})$ which have zero mean. In the same way we consider the spaces $C^r_{m0}(\mathbb{T})$, $r \geq 1$, and the Hölder spaces $C^\theta_{m0}(\mathbb{T})$, $0 < \theta < 1$.
An interesting rigidity result for two-sided ergodic sums by J-P. Conze :
**L** .: *Let $f \in C_{m0}(\mathbb{T})$, $T$ be an irrational rotation. If for some $x_0 \in \mathbb{T}$*

$$\forall n \geq 1, \forall x \in \mathbb{T}, \qquad \sum_{k=-n}^{n-1} T^k f(x) \leq \sum_{k=-n}^{n-1} T^k f(x_0) + C,$$

*then there exists $g \in C(\mathbb{T})$ such that $f = g - Tg$.*

One deduces that continuous functions with a maximizing point $x_0$ and presenting a symmetry with respect to $x_0$ show similar behaviour.

**Cor**.: *Let $f \in C_{m0}(\mathbb{T})$ have a maximizing point $x_0$. If $f(x_0 + x) = f(x_0 - x)$ for all $x \in \mathbb{T}$, then $f = g - Tg$ for some $g \in C(\mathbb{T})$.*

Recall that if $f = g - Tg$ for a measurable $g$ then by ergodicity $g$ is unique up to an additive constant and a null set.

For Hölder continuous functions we have the following result :

**T** .: *Let* $Tx = x + \alpha \mod (1)$ *on* $\mathbb{T}$, $\alpha \notin \mathbb{Q}$. *Any of the following mutually excluding conditions is realized by at least one* $f \in \cap_{0<\theta<1} C^{\theta}_{m0}(\mathbb{T})$.

*i) The point 0 is exactly maximizing for* $f$*, that is,*

$$\forall x \in \mathbb{T},\ \forall n \geq 0, \quad f_n(x) \leq f_n(0)$$

*and there exists* $g \in \bigcap_{1<p<\infty} L^p(\mathbb{T}) \backslash L^\infty(\mathbb{T})$ *such that* $f = g - Tg$*, a.e.*

*In particular,* $g$ *is not continuous.*

*ii) The point 0 is exactly maximizing for* $f$

$$\forall x \in \mathbb{T},\ \forall n \geq 0, \quad f_n(x) \leq f_n(0)$$

*and the skew-product* $(\mathbb{T} \times \mathbb{R}, T_f, \lambda_{\mathbb{T}} \otimes \lambda_{\mathbb{R}})$ *defined by* $f$ *is ergodic, where* $T_f(x, y) = (Tx, y + f(x))$.

*In particular* $f$ *is not a measurable coboundary.*

$(f = g - Tg,\ \Phi(x, y) = g(x) + y,\ \Phi(T_f(x, y)) = g(Tx) + y + g(x) - g(Tx) = \Phi(x, y))$

*iii) Let* $\varepsilon(n) \searrow 0$ *as* $n \nearrow +\infty$. *For any* $x \in \mathbb{T}$ *for a.e* $y \in \mathbb{T}$

$$\sup_{n \in \mathbb{N}} \left\{ n^{-\varepsilon(n)} \Big( f_n(y) - f_n(x) \Big) \right\} = +\infty.$$

*In particular* $f$ *does not have any* $\lambda$*-weakly maximizing point.*

Generic functions in $C(\mathbb{T})$ were also considered.

**T** .: *Let* $Tx = x + \alpha \mod (1)$ *on* $\mathbb{T}$*, with* $\alpha \notin \mathbb{Q}$*. Then a generic function in* $C(\mathbb{T})$ *has no weakly maximizing point.*

Our techniques also allow to treat the case of a dynamical system with a very different nature.

**T** .: *Let* $Tx = 2x \mod (1)$ *on* $\mathbb{T}$*. Then a generic function in* $C(\mathbb{T})$ *has no maximizing point.*

# Fast and slow points of Birkhoff sums

Based on a joint paper with: Frédéric Bayart and Yanick Heurteaux

$\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$,
$C_0(\mathbb{T})$ is the set of continuous functions on $\mathbb{T}$ with zero mean,
topology is generated by the $sup$-norm,
$S_{n,\alpha}f(x)$ is the $n$-th Birkhoff sum, $S_{n,\alpha}f(x) = \sum_{k=0}^{n-1} f(x+k\alpha)$.

$R_\alpha : x \mapsto x+\alpha$ is a uniquely ergodic transformation on $\mathbb{T}$ with respect to the (normalized) Lebesgue measure $\lambda$.
$\Rightarrow$ For all $f \in \mathcal{C}_0(\mathbb{T})$,
by the Ergodic Theorem $\frac{1}{n}\sum_{k=0}^{n-1} f(x+k\alpha) \to 0 = \int_{\mathbb{T}} f d\lambda$,
for $\lambda$-a.e. $x$ and by Weyl's Thm. for all $x$
that is $S_{n,\alpha}f(x) = o(n)$ for all $x \in \mathbb{T}$.

One can fix $x$ and ask for the size of $S_{n,\alpha}f(x)$
for a large proportion of $n \leq N$.
Or one can also fix $n$ and ask for the size of $S_{n,\alpha}f(x)$
for a large set of initial conditions $x$.
Results in these directions are by (for example):
Kesten 1960-61, Beck 2010-11, Huveneers 2009, Bromberg-Ulcigrai 2018.

Our purpose is quite different.
We want to investigate the typical growth of $S_{n,\alpha}f(x)$.

Settings:

• We can fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ (resp. $x \in \mathbb{T}$) and ask for the behaviour of $S_{n,\alpha}f(x)$ for $f$ in a generic subset of $\mathcal{C}_0(\mathbb{T})$ and for a typical $x \in \mathbb{T}$ (resp. for a typical $\alpha \in \mathbb{T}$).

• We can also consider it as a problem of two variables and ask for the behaviour of $S_{n,\alpha}f(x)$ for $f$ in a generic subset of $\mathcal{C}_0(\mathbb{T})$ and for a typical $(\alpha, x) \in \mathbb{T}^2$.

There are also several ways to understand the word "typical".
We can look for a residual set of the parameter space or
for a set of full Lebesgue measure.

( residual = complement of a set of first (Baire) category/meagre set.)

More general setup:

$\Omega$ is an infinite compact metric space

$T : \Omega \to \Omega$ is a uniquely ergodic invertible continuous map.

$\mu$ is the ergodic measure.

We assume that it has full support

(equivalenty, that all orbits of $T$ are dense).

For $x \in \Omega$ and $f \in \mathcal{C}_0(\Omega)$,

the Birkhoff sum $S_{n,T}f(x)$ is now defined by $\sum_{k=0}^{n-1} f\left(T^k x\right)$.

Using $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n) = o(n)$ for $f \in \mathcal{C}_0(\Omega)$ we let

$$\mathcal{E}_\psi(f) = \left\{ x \in \Omega;\ \limsup_n \frac{|S_{n,T}f(x)|}{\psi(n)} = +\infty \right\}.$$

$\mathcal{E}_\psi(f)$ has already been studied by several authors.

In particular, it was shown by Krengel 1978 (when $\Omega = [0,1]$) and later by Liardet and Volný 1997 that,

for all functions $f$ in a residual subset of $\mathcal{C}_0(\Omega)$, $\mu\left(\mathcal{E}_\psi(f)\right) = 1$.

We complete this result by showing that

$\mathcal{E}_\psi(f)$ is also residual (see next slide).

Recall: $\mathcal{E}_\psi(f) = \left\{ x \in \Omega;\ \limsup_n \frac{|S_{n,T} f(x)|}{\psi(n)} = +\infty \right\}.$

**T**.: ★ *Suppose that* $\psi : \mathbb{N} \to \mathbb{N}$ *satisfies* $\psi(n) = o(n)$*.*
*There exists a residual set* $\mathcal{R} \subset \mathcal{C}_0(\Omega)$ *such that*
*for any* $f \in \mathcal{R}$*,* $\mathcal{E}_\psi(f)$ *is residual and of full* $\mu$*-measure in* $\Omega$*.*

**Rem**.: It is important to remark that the conclusion of Theorem ★ deeply depends on the space of functions we consider and on its topology.
For example, in the context of irrational rotations,
Herman showed that $C^{1+\varepsilon}$ zero mean functions are coboundaries,
and their Birkhoff sums are bounded.
For functions of bounded variation, the growth of Birkhoff sums can be estimated very precisely using Koksma's inequality (see next slide).

Koksma's inequality:

**T**.: *Suppose $f : \mathbb{T} \to \mathbb{R}$ is of bounded variation,* Var$(f)$ *and the sequence $\{x_1, ..., x_N\}$ has discrepancy $D_N^*$.*
*Then*

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(x_i) - \int_{\mathbb{T}} f(T) dt \right| \leq \text{Var}(f) D_N^*.$$

Recall:
The discrepancy of the sequence $\{x_1, ..., x_N\}$ is defined by

$$D_N^* = \sup_{I \subset \mathbb{T}} \left| \frac{\text{card}\{1 \leq i \leq N;\ x_i \in I\}}{N} - |I| \right|,\ \text{where } I \text{ is a subinterval of } \mathbb{T}.$$

We could also replace here (and at many places later) the lim sup by lim inf.

**Open Q.:** Let $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n) = o(n)$ and $\lim_{n\to+\infty} \psi(n) = +\infty$.
Does there exist $f \in \mathcal{C}_0(\Omega)$ such that
$\{x \in \Omega;\ \liminf_n |S_{n,T}f(x)|/|\psi(n)| = +\infty\}$ is residual?

The referee suggested to us that if we replace "residual"
by "non negligible set" in this question, the answer is negative.
One can prove that for any $f \in \mathcal{C}_0(\Omega)$, the set

$$A = \{x \in \Omega;\ \liminf_{n\to+\infty} |S_{n,T}f(x)| = +\infty\}$$

is $\mu$-negligible.
This is a consequence of a result due to Atkinson 1976.

$(G,+)$ is a compact and connected metric abelian group.
Then $G$ is a monothetic group, i.e. it possesses a dense cyclic subgroup.
$\mu$ is the Haar measure on $G$.
Group rotations $T_u(x) = x + u$.
$G_0$ is the set of $u \in G$ such that $T_u$ is ergodic.
Well-known results of ergodic theory $\Rightarrow$
$u$ belongs to $G_0$ if and only if $\{nu;\ n \in \mathbb{Z}\}$ is dense in $G$,
in this case $T_u$ is uniquely ergodic,
only the Haar measure is invariant with respect to $T_u$.
$G_0$ is always nonempty, it is dense and its Haar measure equals 1.

Contrary to what happens in Theorem ★ the growth of
$S_{n,u}f(x) = \sum_{k=0}^{n-1} f(x+ku)$ for a typical $(u,x) \in G^2$ is
not the same from the topological and from the probabilistic points of view.

( Recall: $\mathcal{E}_\psi(f) = \left\{ x \in \Omega;\ \limsup_n \frac{|S_{n,T}f(x)|}{\psi(n)} = +\infty \right\}$.

**T**.: ★ *Suppose that* $\psi : \mathbb{N} \to \mathbb{N}$ *satisfies* $\psi(n) = o(n)$. *There exists a residual set* $\mathcal{R} \subset \mathcal{C}_0(\Omega)$ *such that for any* $f \in \mathcal{R}$, $\mathcal{E}_\psi(f)$ *is residual and of full* $\mu$*-measure in* $\Omega$.
)

For the prob./meas. case the typical growth of $S_{n,u}f(x)$ has order $n^{1/2}$:

**T.:** ★★

- *For all $\nu > 1/2$ and all $f \in L_0^2(G)$,*

$$\mu \otimes \mu \left( \left\{ (u,x) \in G^2;\ \limsup_n \frac{|S_{n,u}f(x)|}{n^\nu} \geq 1 \right\} \right) = 0.$$

- *There exists a residual subset $\mathcal{R} \subset \mathcal{C}_0(G)$ such that, for all $f \in \mathcal{R}$,*

$$\mu \otimes \mu \left( \left\{ (u,x) \in G^2;\ \limsup_n \frac{|S_{n,u}f(x)|}{n^{1/2}} = +\infty \right\} \right) = 1.$$

The next step would be to perform a multifractal analysis of the exceptional sets.

Suppose $G = \mathbb{T}$. Let $f \in \mathcal{C}_0(\mathbb{T})$ and $\nu \in (1/2, 1)$.

Set $$\mathcal{E}^-(\nu, f) = \left\{ (\alpha, x) \in \mathbb{T}^2;\ \limsup_n \frac{\log |S_{n,\alpha}f(x)|}{\log n} \geq \nu \right\}.$$

These sets have Lebesgue measure zero.

**Open Q.:** Can we majorize the Hausdorff dimension of $\mathcal{E}^-(\nu, f)$?

From a topological point of view,
the typical growth of $S_{n,u}f(x)$ has order $n$.
Suppose $\psi : \mathbb{N} \to \mathbb{N}$ and

$$\mathfrak{E}_\psi(f) = \left\{ (u,x) \in G^2 \ ; \limsup_n \frac{|S_{n,u}f(x)|}{\psi(n)} = +\infty \right\}.$$

**T**.: ★★★ *Suppose that $\psi : \mathbb{N} \to \mathbb{N}$ satisfies $\psi(n) = o(n)$.*
*There exists a residual set $\mathcal{R}^* \subset \mathcal{C}_0(G) \times G^2$ such that*
*for any $(f,u,x) \in \mathcal{R}^*$ we have $(u,x) \in \mathfrak{E}_\psi(f)$.*

Kuratowski-Ulam theorem and Theorem ★★★ $\Rightarrow$
there exists a residual set $\mathcal{R} \subset \mathcal{C}_0(G)$ such that,
for every $f \in \mathcal{R}$, the set $\mathfrak{E}_\psi(f)$ is residual in $G^2$.

The last possibility is to fix $x \in G$ and allow $u$ to vary.
Without loss of generality, we may assume that $x = 0$.
"Topologically speaking" the typical growth of $S_{n,u}f(0)$ is not better than $o(n)$:

**Cor.**: ♣ *Suppose that $\psi : \mathbb{N} \to \mathbb{N}$ satisfies $\psi(n) = o(n)$. There exists a residual set $\mathcal{R} \subset \mathcal{C}_0(G)$ such that for any $f \in \mathcal{R}$, the set $\{u \in G;\ (u,0) \in \mathfrak{E}_\psi(f)\}$ is residual in $G$.*

Recall: $$\mathfrak{E}_\psi(f) = \left\{(u,x) \in G^2\ ; \limsup_n \frac{|S_{n,u}f(x)|}{\psi(n)} = +\infty\right\}.$$

The measure version:

**Open Q.:** Does there exist $\nu \in [1/2, 1]$ such that
(i) for all $\gamma > \nu$, for all $f \in \mathcal{C}_0(G)$,

$$\mu\left(\left\{u \in G;\ \limsup_n \frac{S_{n,u}f(0)}{n^\gamma} \geq 1\right\}\right) = 0;$$

(ii) for all $\gamma < \nu$, there exists a residual subset $\mathcal{R}$ of $\mathcal{C}_0(G)$ such that,
for all $f \in \mathcal{R}$, $$\mu\left(\left\{u \in G;\ \limsup_n \frac{S_{n,u}f(0)}{n^\gamma} = +\infty\right\}\right) = 1?$$

It can be shown that $\nu = 1/2$ works for (ii).

For irrational rotations we get more precise statements.
Let us fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and set

$$\mathcal{F}_\psi(f) = \left\{x \in \mathbb{T};\ \limsup_n \frac{|S_{n,\alpha}f(x)|}{\psi(n)} < +\infty\right\}.$$

When $\psi(n) = n^\nu$, $\nu \in (0,1)$, we simply denote by $\mathcal{F}_\nu(f)$ the set $\mathcal{F}_\psi(f)$.

By the results mentioned before Theorem ★ :
$\lambda\left(\mathcal{F}_\psi(f)\right) = 0$ for $f$ in a residual subset of $\mathcal{C}_0(\mathbb{T})$,
where $\lambda$ is the Lebesgue measure on $\mathbb{T}$.

This is much stronger:
**T.**: ♢ *For any $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n) = o(n)$,
there exists a residual subset $\mathcal{R}$ of $\mathcal{C}_0(\mathbb{T})$ such that,
for any $f \in \mathcal{R}$,* $\dim_{\mathcal{H}}(\mathcal{F}_\psi(f)) = 0$.

There are also enhancements of meager sets,
for instance $\sigma$-porous sets:
**Open Q.:** Does there exist a residual subset $\mathcal{R}$ of $\mathcal{C}_0(\mathbb{T})$ such that,
for any $f \in \mathcal{R}$, $\mathcal{F}_\psi(f)$ is $\sigma$-porous?

Results for Hölder functions $f \in \mathcal{C}_0^{\xi}(\mathbb{T})$, $\xi \in (0,1)$.

$f$ belongs to $\mathcal{C}_0^{\xi}(\mathbb{T})$ if it has zero mean and
if there exists a constant $C > 0$ such that, for all $x, y \in \mathbb{T}$,
$|f(x) - f(y)| \leq C|x - y|^{\xi}$.
The infimum of such constants $C$ is denoted by $\mathrm{Lip}_{\xi}(f)$.

For a function $f \in \mathcal{C}_0^{\xi}(\mathbb{T})$, we have better bounds on $S_{n,\alpha}f(x)$ depending on $\xi$ and on the arithmetical properties of $\alpha$.

Koksma type inequalities $\Rightarrow$
$|S_{n,\alpha}f(x)| \leq n \cdot \mathrm{Lip}_{\xi}(f)\,(D_n^*(\alpha))^{\xi}$

where $D_n^*(\alpha)$ is the discrepancy of the sequence $(\alpha, 2\alpha, \ldots, n\alpha)$:

$$|D_n^*(\alpha)| = \sup_{I \subset \mathbb{T}} \left| \frac{\mathrm{card}\{1 \leq i \leq n;\ i\alpha \in I\}}{n} - |I| \right|,$$ where $I$ is a subinterval of $\mathbb{T}$.

Recall:

$$\mathcal{F}_\nu(f) = \left\{ x \in \mathbb{T};\ \limsup_n \frac{|S_{n,\alpha} f(x)|}{n^\nu} < +\infty \right\}.$$

If $\alpha$ has type 1 (for example, if $\alpha$ is an irrational algebraic number), well-known estimates of the discrepancy $\Rightarrow$

$|S_{n,\alpha} f(x)| = O(n^{1-\xi+\varepsilon})$ for all $\varepsilon > 0$.

In other words, for all $\nu > 1 - \xi$, $\mathcal{F}_\nu(f) = \mathbb{T}$.

For $\nu \leq 1 - \xi$ the Hausdorff dimension of $\mathcal{F}_\nu(f)$ cannot always be large:

**T**.: ♡ *Let $\xi \in (0,1)$. There exists $f \in \mathcal{C}_0^\xi(\mathbb{T})$ such that, for all $\nu \in (0, 1-\xi)$,* $\dim_{\mathcal{H}}(\mathcal{F}_\nu(f)) \leq \sqrt{\dfrac{\xi}{1-\nu}}$.

This theorem is in stark contrast with a result of Fan and Schmeling 2003.
They study fast Birkhoff averages of subshifts.
In this case, the sets which correspond to $\mathcal{F}_\nu(f)$ always have maximal dimension.

Recall:

**T**.: ♡ *Let* $\xi \in (0,1)$. *There exists* $f \in \mathcal{C}_0^{\xi}(\mathbb{T})$ *such that,*

*for all* $\nu \in (0, 1-\xi)$, $\dim_{\mathcal{H}}(\mathcal{F}_\nu(f)) \leq \sqrt{\frac{\xi}{1-\nu}}$.

We prove slightly more.

Let $\mathcal{E}^{\xi}$ be the closed subspace of $\mathcal{C}_0^{\xi}(\mathbb{T})$ defined by

$$\mathcal{E}^{\xi} = \left\{ f \in \mathcal{C}_0(\mathbb{T});\ \forall x, y \in \mathbb{T},\ |f(x) - f(y)| \leq |x-y|^{\xi} \right\}$$

$$= \left\{ f \in \mathcal{C}_0(\mathbb{T});\ \mathrm{Lip}_{\xi}(f) \leq 1 \right\}.$$

The space $\mathcal{E}^{\xi}$, equipped with the norm of the uniform convergence is now again a separable complete metric space.

We prove:

**T**.: For all functions $f$ in a residual subset of $\mathcal{E}^{\xi}$,

for all $\nu \in (0, 1-\xi)$, $\dim_{\mathcal{H}}(\mathcal{F}_\nu(f)) \leq \sqrt{\frac{\xi}{1-\nu}}$.

**Open Q.:** Is the value $\sqrt{\frac{1-\xi}{\nu}}$ optimal? In particular, it does not depend on the type of $\alpha$, which may look surprizing.

## Coboundaries in $\mathcal{C}_0^\xi(\mathbb{T})$

The natural norm in $\mathcal{C}_0^\xi(\mathbb{T})$ is given by

$$\|f\|_\xi = \sup_{x\in\mathbb{T}} |f(x)| + \sup_{\substack{x,y\in\mathbb{T}\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\xi}.$$

One may wonder whether, in Theorem ♡ we have residuality in $(\mathcal{C}_0^\xi(\mathbb{T}), \| \ \|_\xi)$ instead of in $(\mathcal{E}^\xi, \| \ \|_\infty)$.
A natural way to do that would be to prove that the coboundaries are dense in $\mathcal{C}_0^\xi(\mathbb{T})$.
This is not the case, which shows again that $\mathcal{C}_0^\xi(\mathbb{T})$ is a weird space.
In $\mathcal{C}_0^\xi(\mathbb{T})$ we denote the ball of radius $r$ centered at $f \in \mathcal{C}_0^\xi(\mathbb{T})$ by $B_0^\xi(f,r)$, that is $g \in B_0^\xi(f,r)$ if and only if $\|g-f\|_\xi < r$.

**T.**: ♠ *For any $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ for any $\xi \in (0,1)$ there exists $f \in \mathcal{C}_0^\xi(\mathbb{T})$ such that for any $g \in B_0^\xi(f, 0.1)$ the function $g$ is not a $\mathcal{C}_0$ (and hence not a $\mathcal{C}_0^\xi$)-coboundary, that is there is no $u \in \mathcal{C}_0(\mathbb{T})$ such that $g = u\circ R_\alpha - u$. Hence $\mathcal{C}_0$-coboundaries are not dense in $\mathcal{C}_0^\xi(\mathbb{T})$.*

On the other hand to prove theorems

**T**.: ★ *Suppose that* $\psi : \mathbb{N} \to \mathbb{N}$ *satisfies* $\psi(n) = o(n)$*.*
*There exists a residual set* $\mathcal{R} \subset \mathcal{C}_0(\Omega)$ *such that*
*for any* $f \in \mathcal{R}$*,* $\mathcal{E}_\psi(f)$ *is residual and of full* $\mu$*-measure in* $\Omega$*.*

and

**T**.: ★★★ *Suppose that* $\psi : \mathbb{N} \to \mathbb{N}$ *satisfies* $\psi(n) = o(n)$*.*
*There exists a residual set* $\mathcal{R}^* \subset \mathcal{C}_0(G) \times G^2$ *such that*
*for any* $(f, u, x) \in \mathcal{R}^*$ *we have* $(u, x) \in \mathfrak{E}_\psi(f)$*.*

We use that in the usual $C_0$ topologies
if $T$ is a uniquely ergodic transformation on $\Omega$,
then the set of $\mathcal{C}_0(\Omega)$-coboundaries for $T$,
namely the set of functions $g - g \circ T$ for some $g \in \mathcal{C}_0(\Omega)$,
is dense in $\mathcal{C}_0(\Omega)$.
(see for instance Liardet, P. and Volný, D. 1997).

It is convenient to work with a coboundary
since its Birkhoff sums are uniformly bounded.