Renormalization in Lorenz maps - completely invariant sets and periodic orbits

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18 czerwca 2021

The 9^{th} Visegrad Conference Dynamical Systems, Prague 2021

The talk is based on joint work with Piotr Oprocha

Ł. Cholewa, P. Oprocha, *Renormalization in Lorenz maps – completely invariant sets and periodic orbits*, preprint, arXiv:2104.00110.

Presentation plan

Plan

- Introduction
- Main motivation - Two theorems of Yiming Ding
- Examples and results related to Ding's theorems

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Remark

The last condition implies that the set $\bigcup_{n \in \mathbb{N}_0} f^{-n}(c)$ is dense in $[0,1]$.

A brief historical overview

334 Gerhard Keller and Matthias St. Pierre

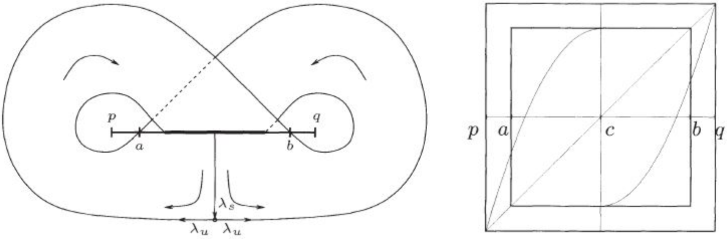

Fig. 1. The left hand side shows a phase portrait of the flow on the branched manifold and the right hand side the first return map to the cross section $\Sigma = [p, q]$.

- In 1976, Guckenheimer proposed a two-dimensional model for the flow on branched manifold, the so-called geometric Lorenz attractor.

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- See more details: P. Raith, *Continuity of the Hausdorff dimension for piecewise monotonic maps*. Israel J. Math. **80** (1992), 97–133.

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- We say that f is a Lorenz map on $[a, b]$, if taking the linear increasing homeomorphism $h\colon [a, b] \to [0, 1]$ the composition $h \circ f \circ h^{-1}$ is a Lorenz map on $[0, 1]$.

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Definition

Let f be an expanding Lorenz map. If there is a proper subinterval $(u, v) \ni c$ of $(0, 1)$ and integers $l, r > 1$ such that the map $g\colon [u, v] \to [u, v]$ defined by

$$g(x) = \begin{cases} f^l(x), & \text{if } x \in [u, c), \\ f^r(x), & \text{if } x \in (c, v], \end{cases}$$

is itself a Lorenz map on $[u, v]$, then we say that f is renormalizable or that g is a renormalization of f and write shortly $g = (f^l, f^r)$.

Definition

We say that $g = (f^m, f^k)$ *is a* *minimal* *renormalization map of an expanding Lorenz map* f, *if any other renormalization* $\tilde{g} = (f^s, f^t)$ *of* f *satisfies* $s \geqslant m$, $t \geqslant k$.

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Definition

A nonempty set $E \subset [0, 1]$ *is said to be completely invariant under* f, *if* $f(E) = E = f^{-1}(E)$.

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Theorem (Ding, 2011)

- *suppose E is a proper completely invariant closed set of f, put*

$$e_- = \sup\{x \in E, x < c\}, \qquad e_+ = \inf\{x \in E, x > c\},$$

$$l = N((e_-, c)), \qquad r = N((c, e_+))$$

Then $f^l(e_-) = e_-$, $f^r(e_+) = e_+$ and the following map

$$R_E f(x) = \begin{cases} f^l(x) & , x \in [f^r(c_+), c) \\ f^r(x) & , x \in (c, f^l(c_-)] \end{cases}$$

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- *if g is a renormalization of f, then there exists a unique proper completely invariant closed set B such that $R_B f = g$.*

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Theorem (Ding, 2011)

Let f be an expanding Lorenz map with minimal period κ, $1 < \kappa < \infty$. Then we have the following statements:

- *f admits a unique κ-periodic orbit O.*
- *$D = \overline{\bigcup_{n=0}^{\infty} f^{-n}(O)}$ is the unique minimal completely invariant closed set of f.*
- *f is renormalizable if and only if $[0,1] \setminus D \neq \emptyset$. If f is renormalizable, then $R_D f$, the renormalization associated to D, is the unique minimal renormalization of f.*
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Let f be a transitive and expanding Lorenz map and let $\hat{f}$ be the map induced on $\mathbb{X}$. Then for every $x \in \mathbb{X}$ the set $\bigcup_{k=0}^{\infty} \hat{f}^{-k}(x)$ is dense in $\mathbb{X}$.

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Corollary

If f is a transitive and expanding Lorenz map without fixed points, then any proper closed set in $[0, 1]$ is not completely invariant for f.

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$$f(x) = \sqrt{2}x + \frac{2-\sqrt{2}}{2} (\text{mod } 1) = \begin{cases} \sqrt{2}x + \frac{2-\sqrt{2}}{2}, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ \sqrt{2}x + \frac{2-\sqrt{2}}{2} - 1, & \text{if } x \in [\frac{1}{2}, 1], \end{cases}.$$

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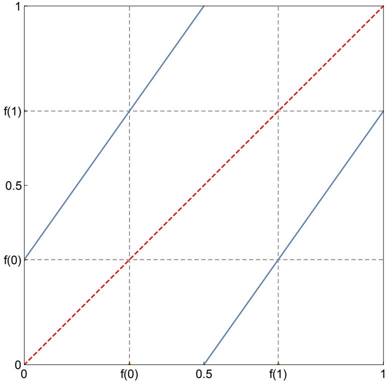
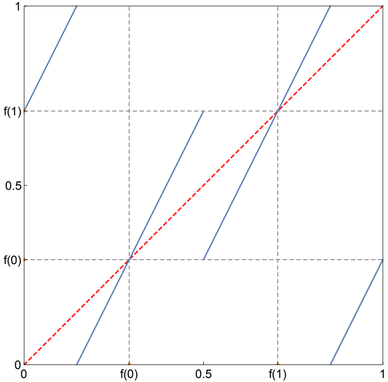
- Note that f is a renormalizable ($g = (f^2, f^2)$) and transitive map.

Example 1 - Graphs of $f(x)$ and $f^2(x)$





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Candidates for a completely invariant sets - Idea of Ding

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Question

When J_g is a proper completely invariant closed subset of $[0,1]$?

Primary $n(k)$-cycles

Definition (Glendinning, 1990)

A periodic orbit $\{z_j = f^j(z_0) : j \in \{0, \dots, n-1\}\}$ of period n of an expanding Lorenz map f is an $n(k)$-cycle if its points satisfy

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then the $n(k)$*-cycle is primary.*

Primary $n(k)$-cycles, renormalizations and completely invariant sets

Theorem (Ch., Oprocha)

Let f be an expanding Lorenz map with a primary $n(k)$-cycle

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Then the following conditions hold:

- *the following $g\colon [u,v] \to [u,v]$ provided below is a well defined expanding Lorenz map which additionally is a renormalization of f:*

$$g(x) = \begin{cases} f^n(x); & x \in [u,c) \\ f^n(x); & x \in (c,v] \end{cases},$$

where $[u,v] := [f^{n-1}(0), f^{n-1}(1)]$.

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Theorem (Ch., Oprocha)

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- *if $z_{k-1} \neq f(0)$ and $z_k \neq f(1)$ then:*
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    - *$z_{n-k-1} = \sup\{x \in J_g, x < c\}$ and $z_{n-k} = \inf\{x \in J_g, x > c\}$.*
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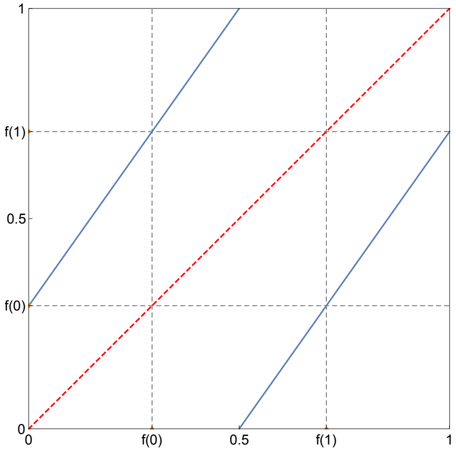

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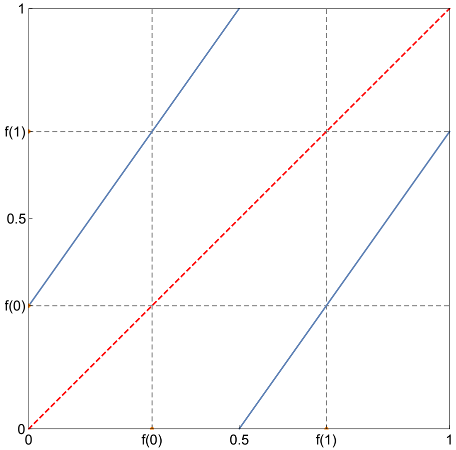

- $O = \{z_0, z_1\} = \{f(0), f(1)\}$ forms a primary 2(1)-cycle

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Example 2 - Three renormalizations with the same set J_g

- Denote

$$W := -8 \cdot \left(\frac{3}{9 + \sqrt{849}} \right)^{\frac{1}{3}} + \left(2 \left(9 + \sqrt{849} \right) \right)^{\frac{1}{3}}$$

$$\beta := \frac{\sqrt{\sqrt{W} + \sqrt{-W + \frac{12}{\sqrt{W}}}}}{2^{\frac{2}{3}} \cdot 3^{\frac{1}{6}}} \approx 1.1048$$

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- The map $f(x) = \beta x + \alpha (\mathrm{mod}\ 1)$ is expanding Lorenz map with critical point $c = \frac{1-\alpha}{\beta} \approx 0.5085$.

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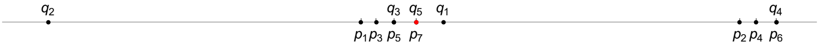

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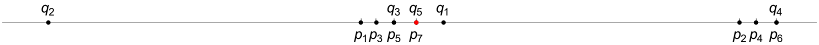

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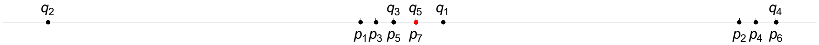

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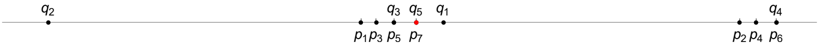

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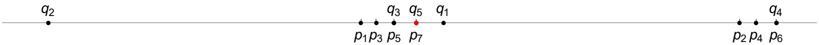
- Hence J_g associated to renormalization $g = (f^2, f^2)$ is a closed, completely invariant and proper subset of $[0, 1]$,
- Observe that

$$[\hat{u}, \hat{v}] := [f^6(c_+), f^8(c_-)] = [f^5(0_+), f(1_-)] = [f^6(c_+), f^2(c_-)]$$

so on $[\hat{u}, \hat{v}]$ we have two well defined renormalizations $\hat{g} = (f^8, f^6)$ and $\overline{g} = (f^2, f^6)$

Example 2 - Three renormalizations with the same set J_g



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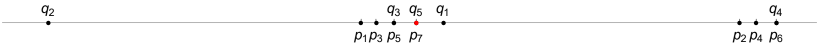

- Clearly $(\hat{u}, \hat{v}) \subset (f(0), f(1)) = (u, v)$, while $f^4((u, \hat{u})) = (\hat{u}, \hat{v})$ and $f^2(\hat{u}) \in (\hat{u}, \hat{v})$,

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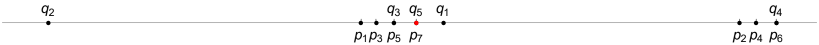

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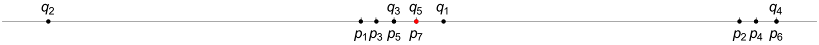

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- Therefore $F_{\hat{g}} = F_g$, so also $J_{\hat{g}} = J_g$
- All three renormalizations define the same completely invariant set J_g, while only g can be recovered from J_g by procedure presented in Ding's Theorem.

Periodic renormalizations

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Theorem (Ch., Oprocha)

Let f be an expanding Lorenz map with a primary $n(k)$-cycle

$$z_0 < z_1 < \cdots < z_{n-k-1} < c < z_{n-k} < \cdots < z_{n-1}$$

such that $z_{k-1} \neq f(0)$ and $z_k \neq f(1)$. Then the renormalization $R_D f$ of f associated to minimal cycle O of f is well defined, periodic and equal to $g = (f^n, f^n)$.

Example 3 - $R_D f$ is not the minimal renormalization

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- Consider an expanding Lorenz map $f\colon [0,1] \to [0,1]$ defined by $f(x) = \beta x + \alpha (\text{mod } 1)$, where

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$$z_0 := \frac{\alpha_0}{1-\beta^5}, \quad \text{where} \quad \alpha_0 := \beta^4\alpha + \beta^3\alpha + \beta^2\alpha + \beta\alpha - \beta^2 + \alpha - 1.$$

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- Then $z_0 \approx 0.11227$ and the orbit $O := Orb(z_0) = \{z_0, z_1, z_2, z_3, z_4\}$ forms a primary 5(2)-cycle for f.

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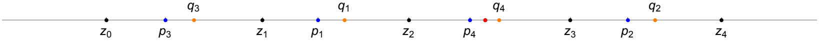

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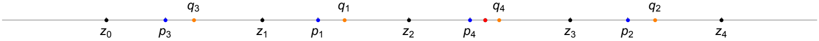

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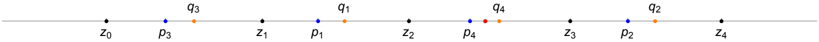

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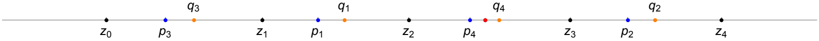

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- But there exists another renormalization $\tilde{g} = (f^3, f^2)$,
- So the renormalization g associated to set D is not the minimal renormalization of f.

Example 3 - $R_D f$ is not the minimal renormalization

Theorem (Ding, 2011)

Let f be an expanding Lorenz map with minimal period κ, $1 < \kappa < \infty$. Then we have the following statements:

- *f admits a unique κ-periodic orbit O.*
- *$D = \overline{\bigcup_{n=0}^{\infty} f^{-n}(O)}$ is the unique minimal completely invariant closed set of f.*
- *f is renormalizable if and only if $[0,1] \setminus D \neq \emptyset$. If f is renormalizable, then $R_D f$, the renormalization associated to D, is the unique minimal renormalization of f.*
- *The following trichotomy holds: (i) $D = [0,1]$, (ii) $D = O$, (iii) D is a Cantor set.*

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The end

Thank you for your attention!

Děkuji za pozornost!