Pseudo-arc as attractor in the disk: topological and measure-theoretical aspects

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Joint work with Piotr Oprocha (AGH Krakow)

Motivation and the first result

Thm (Bing, 1951): For any manifold M of dimension at least 2, the set of subcontinua homeomorphic to the pseudo-arc is a dense G_δ subset of the set of all subcontinua of M.

Thm (Block, Keesling, Uspenskij, 1999): Set of maps $f \in C(I)$ for which $\varprojlim(I, f)$ is pseudo-arc is nowhere dense in $C(I)$.

Thm (Bobok, Troubetzkoy, 2020): Typical map from

$$C_\lambda(I) := \{f \in C(I); \lambda(f^{-1}(A)) = \lambda(A) \text{ for all } A \in \mathcal{B}\}$$

is locally eventually onto (leo),[1] and weakly mixing wrt λ.[2]

Question 1: Does there exist an interval map $f \in C_\lambda(I)$ so that $\varprojlim(I, f)$ is the pseudo-arc?

Thm (Č., Oprocha, 2021) For a typical map $f \in C_\lambda(I)$, $\varprojlim(I, f)$ is the pseudo-arc.

[1] $f : I \to I$ is leo: $\forall$ open interval $J \subset I$ there exists $n \geq 0$ so that $f^n(J) = I$.
[2] if for every $A, B \in \mathcal{B}$, $\lim_{n\to\infty} \sum_{j=0}^{n-1} |\lambda(f^{-j}(A) \cap B) - \lambda(A)\lambda(B)| = 0$.

Introduction

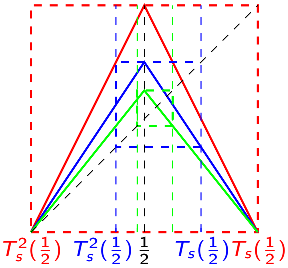
Say X a compact connected metric space (continuum) and $f : X \to X$ a map. Inverse limit space

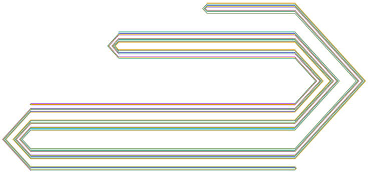
$$\hat{I} := \varprojlim(X, f) = \{(x_0, x_1, x_2, \ldots) \in X^{\infty}; f(x_{i+1}) = x_i\}.$$

The shift homeomorphism (natural extension of f):

$$\hat{f}((x_0, x_1, x_2, \ldots)) = (f(x_0), x_0, x_1, \ldots).$$

E.g.: $T_s(x) = \min_{x \in J := [T_s^2(\frac{1}{2}), T_s(\frac{1}{2})]} \{sx, s(1-x)\}, K_s = \varprojlim(J, T_s).$





Intro to Lebesgue preserving interval maps

Def: By λ we denote the Lebesgue measure on I and $\mathcal{B}$ Borel sets in I. $C_\lambda(I)$ denotes the space of all λ-preserving continuous interval maps; we use $\rho(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|$.

Prop: $(C_\lambda(I), \rho)$ is a complete metric space.

Example: $T_2 \in C_\lambda(I)$ but $T_s \notin C_\lambda(I)$ for all $s \in (1,2)$.

Prop: Let f be piecewise affine with nonzero slopes and such that its derivative does not exist at a finite set E. Then $f \in C_\lambda(I)$ iff

$$\forall\ y \in [0,1) \setminus f(E)\colon \quad \sum_{x\in f^{-1}(y)} \frac{1}{|f'(x)|} = 1. \tag{1}$$

Crookedness and pseudo-arc

Def: Let $f : I \to I$, let $a < b \in I$ and let $\delta > 0$. We say that f is δ-crooked between a and b if, for every two points $c, d \in I$ such that $f(c) = a$ and $f(d) = b$, there are points $c < c' \leq d' < d$ such that $|b - f(c')| \leq \delta$ and $|a - f(d')| \leq \delta$. We will say that f is δ-crooked if it is δ-crooked between every pair of points.

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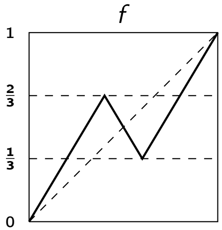

Figure: Map f is $1/3$-crooked.

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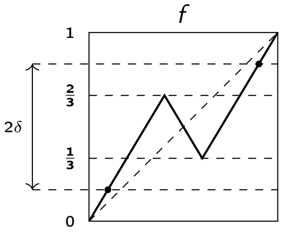

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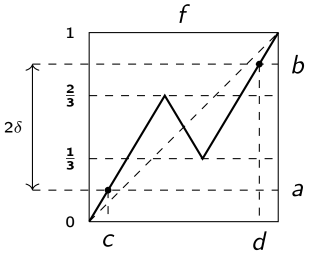

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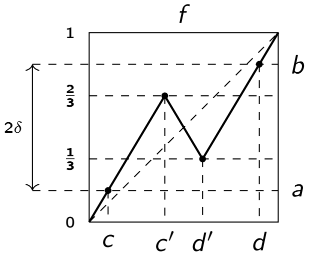

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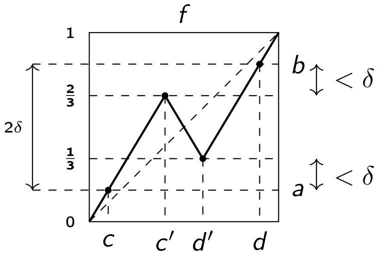

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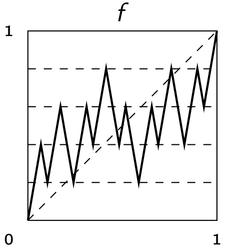
Def: Set $I := [0,1]$. Let $f : I \to I$, let $a < b \in I$ and let $\delta > 0$. We say that f is δ-crooked between a and b if, for every two points $c, d \in I$ such that $f(c) = a$ and $f(d) = b$, there are points $c < c' \leq d' < d$ such that $|b - f(c')| \leq \delta$ and $|a - f(d')| \leq \delta$. We will say that f is δ-crooked if it is δ-crooked between every pair of points.



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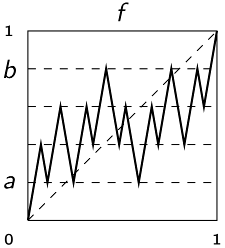

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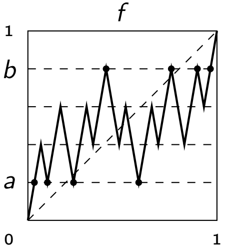

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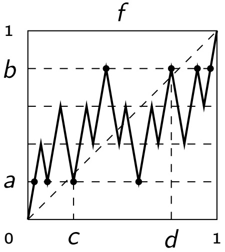

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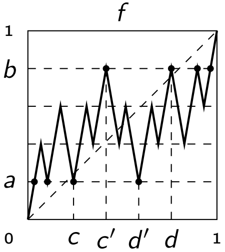

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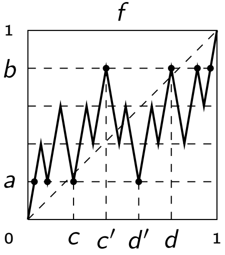

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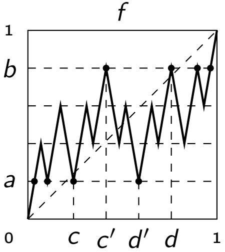
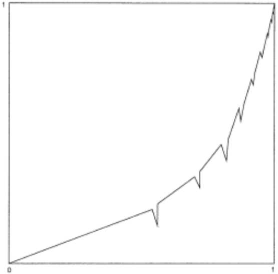


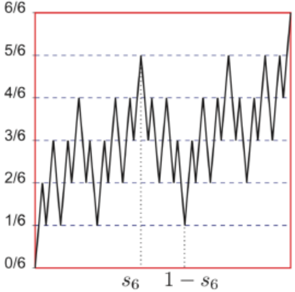
Figure: Map f is $1/5$-crooked.

Lem (Bing 1950's & Minc,Transue, 1991): Nondegenerate continuum $\hat{I}$ is pseudoarc $\iff$ $f : I \to I$ is such that for every $\delta > 0$ there is $n \geq 0$ such that f^n is δ-crooked.

Pseudo-arc

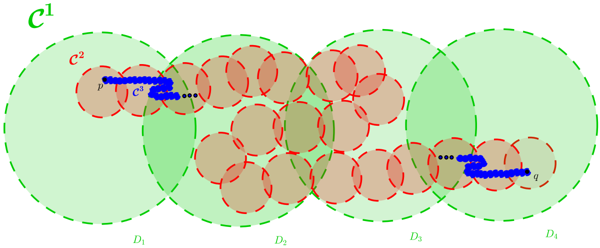


Henderson's interval map (approx.) with 0 topological entropy, such that $\hat{I}$ is a pseudo-arc



Approx. of an interval map with positive topological entropy, such that $\hat{I}$ is a pseudo-arc (Minc& Transue)

Computer simulation of an approx. of pseudo-arc (courtesy of Jan Boroński)



Strategy of proof that δ-crookedness of f^n is typical in $C_\lambda(I)$

- Take any piecewise linear leo map $g \in C_\lambda(I)$ (Bobok, Troubetzkoy 2020 prove they are dense in $C_\lambda(I)$),
- For every $\varepsilon > 0$ there exists $G \in C_\lambda(I)$ such that G is admissible[3] and $|G(t) - g(t)| < \varepsilon$ for every $t \in I$,
- Perturb G with an arbitrary small perturbation $\lambda_{n,k} \in C_\lambda(I)$, i.e. $\tilde{G} = G \circ \lambda_{n,k} \in C_\lambda(I)$,
- Fix $\eta, \delta \in \mathbb{R}^+$ for $\tilde{G}$. There $\exists$ an admissible map $F \in C_\lambda(I)$ and $n \in \mathbb{N}$ s.t. F^n is δ-crooked and $|\tilde{G}(t) - F(t)| < \eta$ for every $t \in I$ (Minc, Transue 1990),
- Let $F, f \in C_\lambda(I)$ be two maps so that $\rho(F, f) < \varepsilon$. If F is δ-crooked, then f is $(\delta + 2\varepsilon)$-crooked (Minc, Transue 1990).

[3] G is admissible if it is leo and $|G'(x)| > 4$ for all $x \in I$ where it is defined.

Assumptions $\lambda_{n,k}$ needs to fulfill

Perturbations $\lambda_{n,k}$ need to satisfy the following. Set $\varepsilon := \frac{n-1}{n+k-1}$ and $\gamma := \frac{1}{n+k-1}$. Then the following statements hold for every odd integer $n \geq 7$ and $k \geq 1$:

(i) $\lambda_{n,k} \in C_\lambda(I)$ has constant slope.

(ii) $|t - \lambda_{n,k}(t)| < \varepsilon/2 + \gamma$ for each $t \in I$,

(iii) for every a and b such that $|a - b| < \varepsilon$, $\lambda_{n,k}$ is γ-crooked between a and b,

(iv) for each subinterval A of I we have $\operatorname{diam}(\lambda_{n,k}(A)) \geq \operatorname{diam}(A)$,

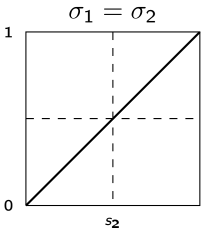
and if, additionally, $\operatorname{diam}(A) > \gamma$, then

(v) $\operatorname{diam}(\lambda_{n,k}(A)) > \varepsilon/2$,

(vi) $A \subset \lambda_{n,k}(A)$ and

(vii) $\lambda_{n,k}(B) \subset B(\lambda_{n,k}(A), r + \gamma)$ for each real number r and each set $B \subset B(A, r)$.

Simple *n*-crooked maps (Lewis&Minc, 2010)



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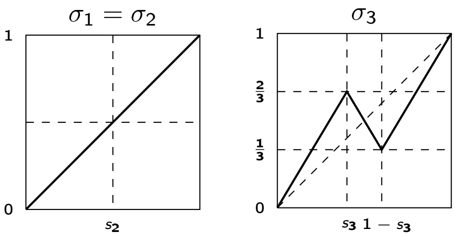

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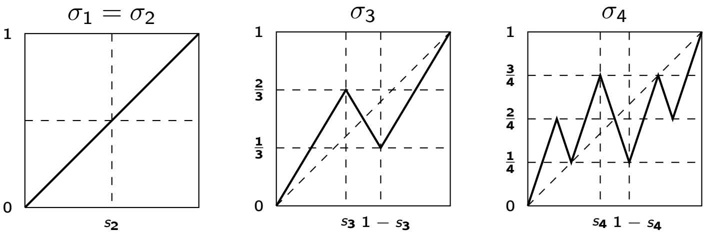

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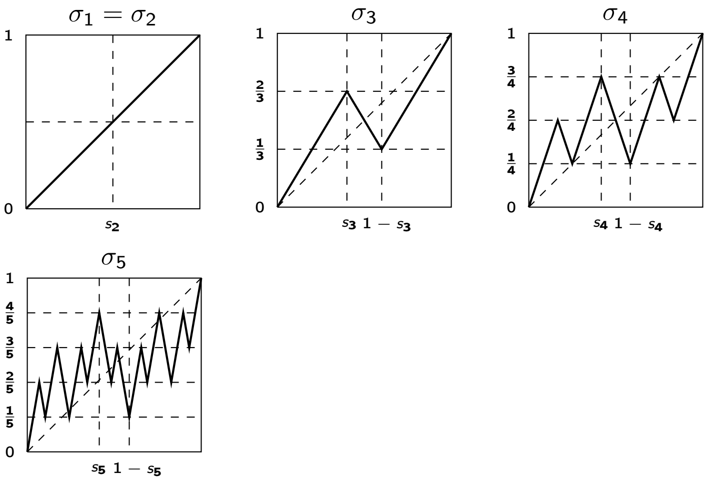

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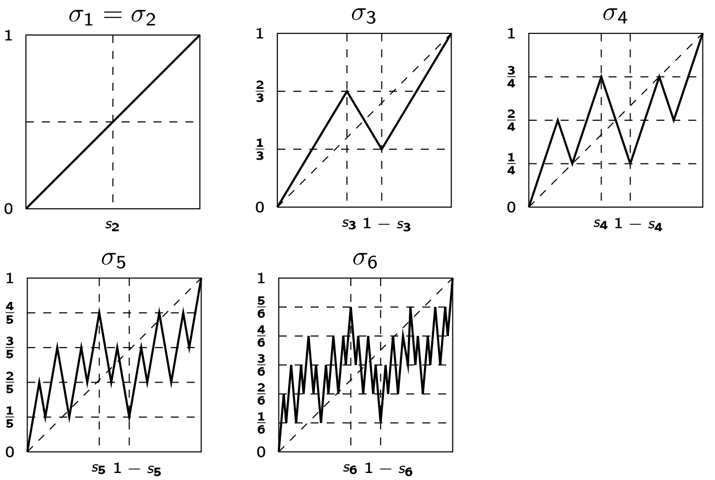

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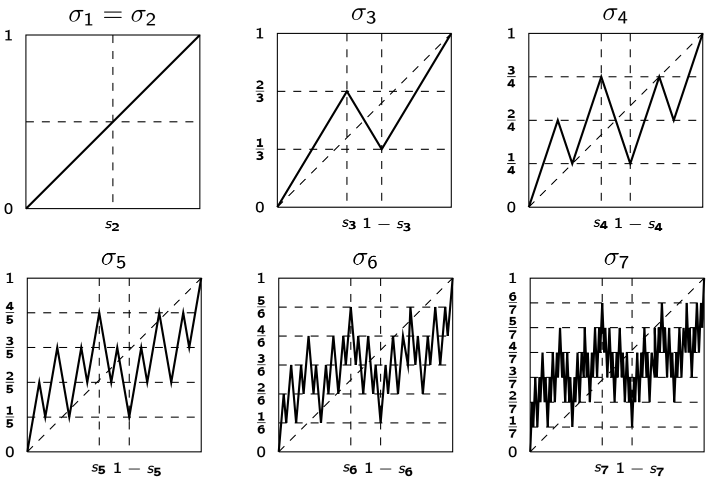


Figure: Simple n-crooked maps σ_n for $n = 1, \ldots 7$.

Building blocks of perturbations $\hat{\lambda}_{n,k}$

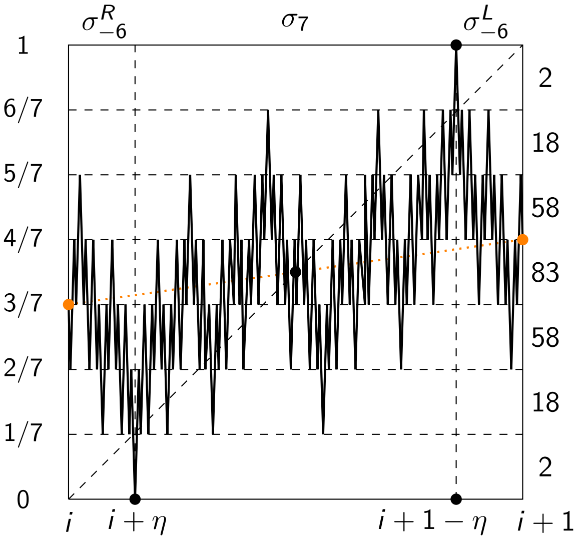
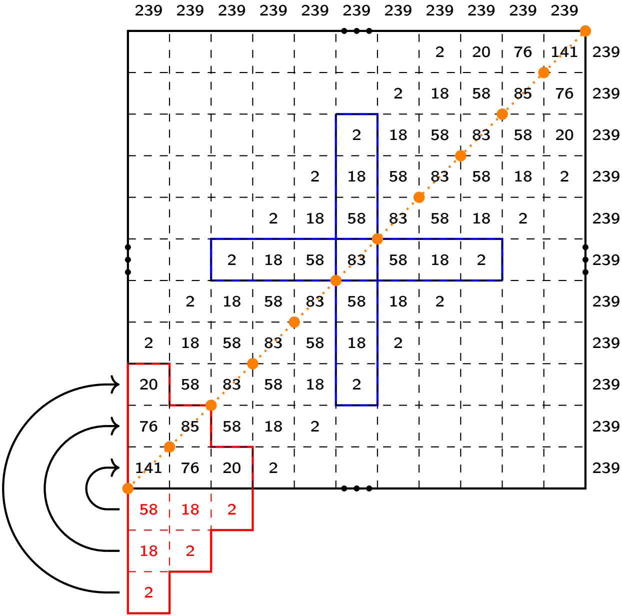


Figure: Building blocks of the function $\hat{\lambda}_{n,k}$ for $n = 7$.

Perturbations $\hat{\lambda}_{n,k}$ and $\lambda_{n,k}$ (with a flip)



Inverse limits and natural extensions revisited

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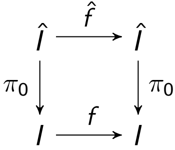
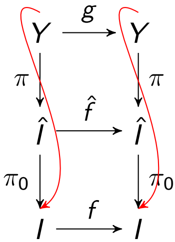
Natural projection $\pi_0 : \hat{I} \to I$ defined by $\pi_0(x) = x_0$ semi-conjugates $\hat{f}$ to f.



Say that $g : Y \to Y$ is an invertible dynamical system and $p : Y \to I$ factors g to f, then p factors through π_0: i.e. $\hat{f}$ is the simplest invertible system which extends f.



$p = \pi_0 \circ \pi$

The second result

Question 2: Without involving measures, can we find a natural space of interval maps for which the typical inverse limit is pseudo-arc?

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Def: Denote by $C_{DP}(I) \subset C(I)$ the class of interval maps with the dense set of periodic points. The space $(\overline{C_{DP}(I)}, \rho)$ is closed in $(C(I), \rho)$ and thus it is a complete space as well.

Obs: Let f be an interval map. The following conditions are equivalent.

(i) f has a dense set of periodic points, i.e., $\overline{\mathrm{Per}(f)} = I$.

(ii) f preserves a nonatomic probability measure μ with $\mathrm{supp}\ \mu = I$.

(iii) There exists a homeomorphism h of I such that $h \circ f \circ h^{-1} \in C_\lambda(I)$.

Thm (Č., Oprocha, 2021) For a typical map $f \in \overline{C_{DP}(I)}$, space $\hat{I}$ is the pseudo-arc.

The third result

There is a prescribed way to make $\hat{I}$ attractor of some o.p. plane homeomorphism using the Brown-Barge-Martin embeddings.

Question 3: Are these planar embeddings of the pseudo-arc maybe all dynamically the same?

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Question 3: Are these planar embeddings of the pseudo-arc maybe all dynamically the same?

Thm (Č, Oprocha, 2021): There is a parameterised family of homeomorphisms $\{\Phi_t\}_{t\in I} \subset \mathcal{H}(D, D)$ with $I \subset \mathcal{T} \subset C_\lambda(I)$ where $\mathcal{T}$ is a dense G_δ set from above such that:

(a) For each $t \in I$ there is a Φ_t-invariant set $\Lambda_t \subset D$ homeomorphic to the pseudo-arc so that:
  (i) $\Phi_t|_{\Lambda_t} : \Lambda_t \to \Lambda_t$ is topologically conjugate to $\hat{f}_t : \hat{I}_t \to \hat{I}_t$.
  (ii) If $x \in D \setminus \partial D$, then the omega limit set $\omega(x, \Phi_t) \subset \Lambda_t$.

(b) The attracting sets $\{\Lambda_t\}_{t\in I}$ vary continuously in Hausdorff metric.

(c) Prime ends rotation numbers of homeomorphisms $\{\Phi_t\}_{t\in I}$ vary continuously in the interval $[0, 1/2]$.

(d) There are countably many dynamically different embeddings of pseudo-arc in the family $\{\Lambda_t\}_{t\in I}$.

The fourth result: measure-theoretical BBMs

Thm(Č., Oprocha, 2021): There is a typical set $\widetilde{\mathcal{T}} \subset C_\lambda(I)$ and a parameterized family of o. p. homeomorphisms $\{\Phi_f\}_{f\in\widetilde{\mathcal{T}}} \subset \mathcal{H}(D,D)$ so that:

(a) for each $f \in \widetilde{\mathcal{T}}$ there is a compact Φ_f-invariant set $P_f \subset D$ homeomorphic to the pseudo-arc so that:

(i) $\Phi_f|_{P_f}$ is topologically conjugate to $\hat{f} : \hat{I}_f \to \hat{I}_f$.

(ii) If $x \in D \setminus \partial D$, then the omega limit set $\omega(x, \Phi_f) \subset P_f$.

(b) the attractors $\{P_f\}_{f\in\widetilde{\mathcal{T}}}$ vary continuously with $f \in \widetilde{\mathcal{T}}$ in the Hausdorff metric.

(c) for each $f \in \widetilde{\mathcal{T}}$ the attractors P_f preserve induced measure μ_f invariant for Φ_f for any $f \in \widetilde{\mathcal{T}}$. Let λ_f be an induced Oxtoby-Ulam measure on D. There exists an open set $U \subset D$ which for each f contains $U_f \subset U$ so that $\lambda_f(U_f) = \lambda(U)$ and U_f is in the basin of attraction of μ_f. In particular each μ_f is physical measure.

(d) there exist a dense countable set of maps $g \in \widetilde{\mathcal{T}}$ for which μ_g is the unique physical measure, i.e. its basin of attraction has the full λ_g-measure in D.

(e) $\Phi_f|_{P_f}$ are topologically mixing, have the shadowing property and are weakly mixing wrt μ_f.

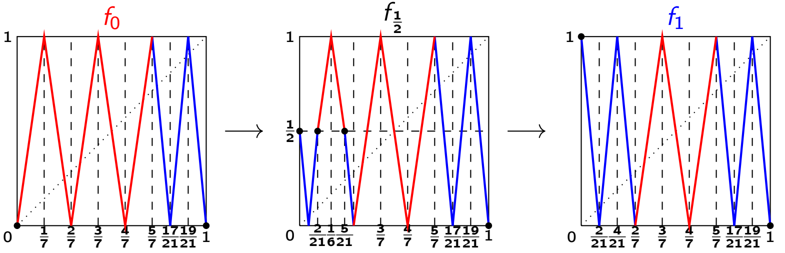
(f) measures μ_f vary continuously in the weak* topology.

Thank you!

A family $\{f_t\}_{t\in I} \subset \mathcal{T}$

For any $t \in I$ let f_t be defined $f_t(\frac{2}{7}) = f_t(\frac{4}{7}) = f_t(\frac{17}{21}) = f_t(1) = 0$ and $f_t(\frac{3}{7}) = f_t(\frac{5}{7}) = f_t(\frac{19}{21}) = 1$ and piecewise linear between these points on the interval $[\frac{2}{7}, 1]$. Furthermore on interval $x \in [0, \frac{2}{7}]$ let:

$$f_t(x) = \begin{cases} 7(x - t\frac{4}{21}); & x \in (1-t)[0, \frac{1}{7}] + t\frac{4}{21}, \\ 1 - 7(x - \frac{1}{7}(1-t) - t\frac{4}{21}); & x \in (1-t)[\frac{1}{7}, \frac{2}{7}] + t\frac{4}{21}, \\ \frac{21}{2}(x - t\frac{2}{21}); & x \in t[\frac{2}{21}, \frac{4}{21}], \\ 1 - \frac{21}{2}x; & x \in t[0, \frac{2}{21}], \\ 1 - \frac{21}{2}(x - t(\frac{2}{7} - t\frac{2}{21})); & x \in t[\frac{2}{7} - t\frac{2}{21}, \frac{2}{7}]. \end{cases}$$



Block, Keesling, Uspenskij argument

Thm (Block, Keesling, Uspenskij, 1999): Set of maps $f \in C(I)$ for which $\varprojlim(I, f)$ is pseudo-arc is nowhere dense in $C(I)$.