Minimal direct products and product-minimal spaces

Matúš Dirbák

(joint work with Ľubomír Snoha and Vladimír Špitalský)

Matej Bel University

Matus.Dirbak@umb.sk

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- G. D. Birkhoff: *Quelques théorèmes sur le mouvement des systèmes dynamiques*, Bulletin de la Société mathématiques de France, **40** (1912), 305-323.

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3. Minimal systems are often viewed as topological analogues of ergodic systems from ergodic theory.
4. Aesthetic reasons — there is certain beauty to this notion and the theory around it.

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(vi) all invariant Borel probability measures of $\mathcal{X}$ have full support.

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Remark 1

We infer from condition (iii) that minimality is a G_δ-property. This suggests the possibility of using Baire category method to verify the existence of minimal maps.

‘Strange’ minimal spaces

Theorem 1 (Downarowicz, Snoha, Tywoniuk, 2015)

There exist one-dimensional continua X with the following properties:

- *the homeomorphism group $\mathcal{H}(X)$ of X is infinite cyclic,*
- *all (non-identical) homeomorphisms on X are minimal,*
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Theorem 2 (Boroński, Činč, Foryś-Krawiec, 2019)

For every $h \in [0, \infty)$ there exists a compact space Z_h with the following properties:

- *Z_h admits a minimal map with topological entropy h,*
- *the homeomorphism group of Z_h is degenerate.*

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- *$Y \times Y$ does not admit any minimal homeomorphism.*

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Theorem 4 (Snoha, Špitalský, 2018)

The spaces X constructed by Downarowicz, Snoha and Tywoniuk have the following properties:

- X *admits a minimal homeomorphism,*
- $X \times X$ *admits no (invertible or non-invertible) minimal maps.*

Product-minimal spaces — motivation

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Theorem 5 (Kolyada, Snoha, Trofimchuk, 2014)

Given an arbitrary minimal system $\mathcal{X}$, there is an irrational rotation $\mathcal{R}$ on $\mathbb{S}^1$ such that the product $\mathcal{X} \times \mathcal{R}$ is minimal.

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Remark 2

Thus, the circle $\mathbb{S}^1$ admits minimal homeomorphisms which are independent, in the sense of disjointness, from an arbitrary given minimal system $\mathcal{X}$.

Product-minimal spaces — definition

Definition 6 (Product-minimality)

A compact metrizable space Y is called *product-minimal* (briefly, PM) if for every minimal system (X, T) there is a continuous map $S\colon Y \to Y$ such that the product $(X, T) \times (Y, S)$ is minimal.

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Remark 3

In this terminology, the circle $\mathbb{S}^1$ is HPM and, of course, also PM.

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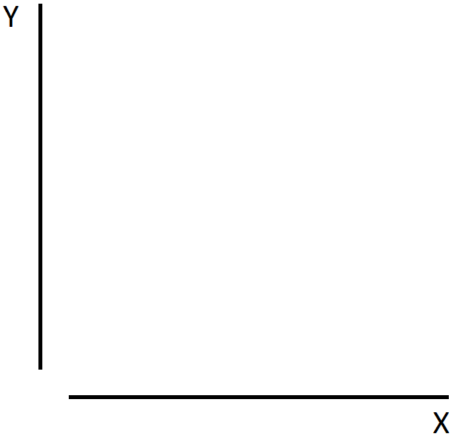

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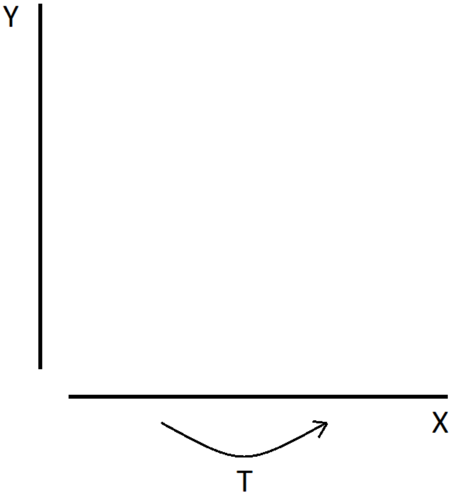

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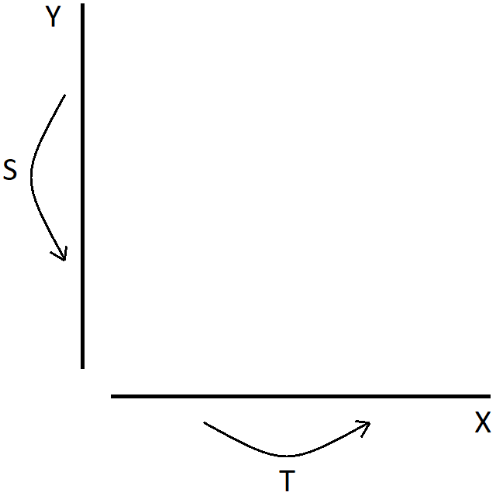

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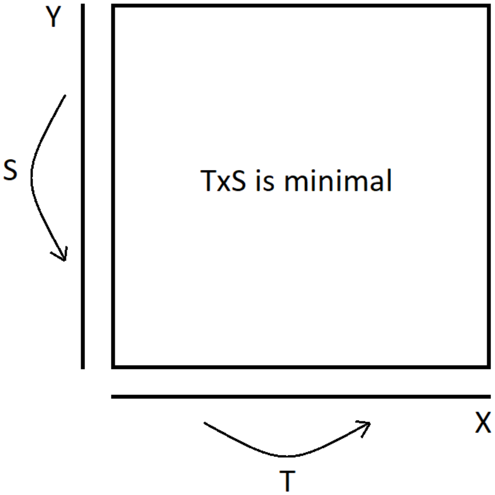

First observations on (H)PM-spaces

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The following statements hold.

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3. *The class of (H)PM-spaces is closed with respect to finite and countably infinite products.*
4. *The family of isomorphism classes of minimal transformations on a nondegenerate PM-space is uncountable. The same is true for isomorphism classes of minimal homeomorphisms on an HPM-space.*

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are surjective.

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This contradicts the assumption that Y is product-minimal. □

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(f) *every odd-dimensional sphere.*

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Thus, we have a measure-preserving action of the multiplicative semigroup $\mathbb{N}^\star = (\mathbb{N}, \cdot)$ on G.

Lemma 9

The action of $\mathbb{N}^\star$ *on* G *described above is mixing in the sense that*

$$\lim_{n\to\infty} \mu\left(A \cap E_n^{-1}(B)\right) = \mu(A)\mu(B) \tag{1}$$

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Formula (1) can be rewritten, in the usual way, as

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Now, characteristic functions of measurable sets generate a dense linear subspace of $L^2(\mu)$. Consequently, our problem translates into showing that

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So let $\gamma, \delta \in G^*$ and recall that

$$\int \varrho\, d\mu = 0 \quad \forall \varrho \in G^*, \varrho \neq 1.$$

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If $\delta = 1$, the equality is true. So let $\delta \neq 1$. Since G is connected, G^* is torsion-free. Consequently, $\gamma \cdot \delta^n = 1$ may occur for at most one n.

Proof.

$$\lim_{n\to\infty} \int \gamma \cdot \delta^n \, d\mu \stackrel{?}{=} \int \gamma \, d\mu \int \delta \, d\mu.$$

If $\delta = 1$, the equality is true. So let $\delta \neq 1$. Since G is connected, G^* is torsion-free. Consequently, $\gamma \cdot \delta^n = 1$ may occur for at most one n. Thus,

$$\int \gamma \cdot \delta^n \, d\mu = 0 = \int \gamma \, d\mu \int \delta \, d\mu$$

for all sufficiently large n. □

Lemma 10

The action of $\mathbb{N}^\star$ *on* G *described above is topologically mixing in the sense that*

$$\forall\, V, W \subseteq G \text{ nonempty open } \exists\, m \forall\, n \geq m \, : \, V \cap E_n^{-1}(W) \neq \emptyset.$$

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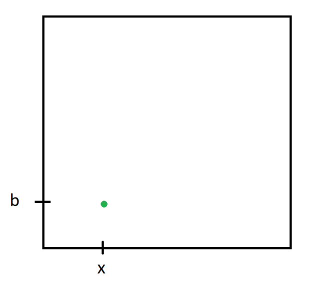
Let (x, b) be a point with a dense orbit.



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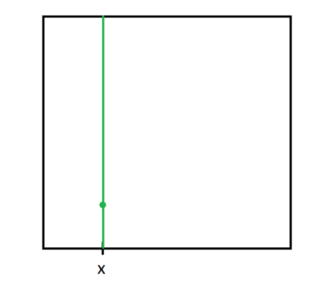
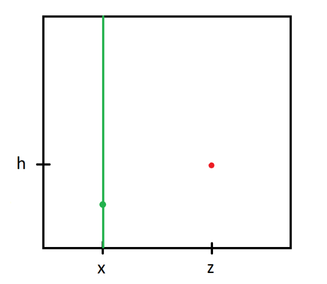


Proof.

To show that $\mathcal{X} \times \mathcal{R}_a$ is minimal, fix arbitrary $(z, h) \in X \times G$.



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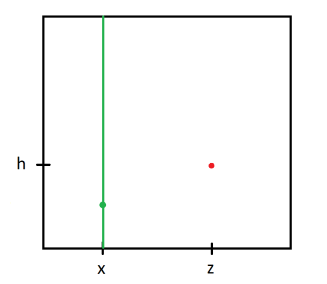

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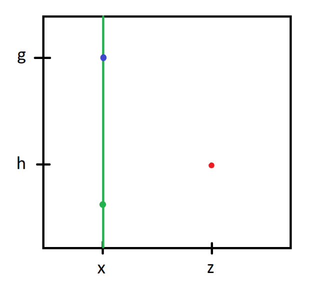

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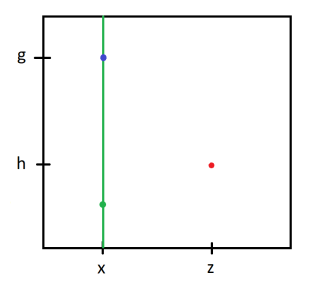

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$\square$

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There is $a \in G$ such that the product system $\mathcal{X} \times \mathcal{R}_a$ is minimal. Thus, G is an HPM-space.

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Now let $\mathcal{B}$ and $\mathcal{C}$ be countable bases of X and G, respectively. By virtue of Lemma 11, the system $\mathcal{X} \times \mathcal{R}_a$ is minimal if, and only if,

$$a \in \bigcap_{U \in \mathcal{B}} \bigcap_{W \in \mathcal{C}} H_{U,W}.$$

Proof.

Consequently, to finish the proof, it is sufficient to show that our sets

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$$\emptyset \neq V \cap E_n^{-1}(W) \subseteq V \cap H_{U,W}.$$

Thus, $H_{U,W}$ is dense, indeed. □

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Open problem — generalized solenoids

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Let Y be a compact metrizable space and let $\phi = (\varphi_t)_{t \in \mathbb{R}}$ be a minimal continuous flow on Y. Consider the centralizer $Z(\phi)$ of ϕ in $\mathcal{H}(Y)$

$$Z(\phi) = \{h \in \mathcal{H}(Y) : h \circ \varphi_t = \varphi_t \circ h \text{ for every } t \in \mathbb{R}\}.$$

If $Z(\phi)$ acts transitively on Y in the algebraic sense then Y is HPM.

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Does the theorem remain true if we drop the assumption of transitive centralizer?

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Question 2

Are all generalized solenoids HPM?

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Remark 4

Every solenoid is a compact connected metrizable abelian group, hence admits a minimal flow with a transitive centralizer.

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Fix $x \in X$ and $y \in Y$. Given a nonempty open set $V \subseteq Y$, the set

$$E_\phi(y, V) = \{t \in \mathbb{R} : \varphi_t(y) \in V\}$$

is syndetic by minimality of ϕ and compactness of Y.

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and the union is dense, since $E_\phi(y,V)$ is syndetic.

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Let (x, y) be a point with a dense orbit.

Proof.

Now choose countable bases $\mathcal{B}$ and $\mathcal{C}$ for X and Y, respectively. Then (x, y) has a dense orbit in $\mathcal{X} \times \mathcal{Y}_t$ if, and only if,

$$t \in \bigcap_{U \in \mathcal{B}} \bigcap_{V \in \mathcal{C}} A_{U,V}.$$

The intersection is the desired residual subset of $\mathbb{R}$. □

Lemma 17

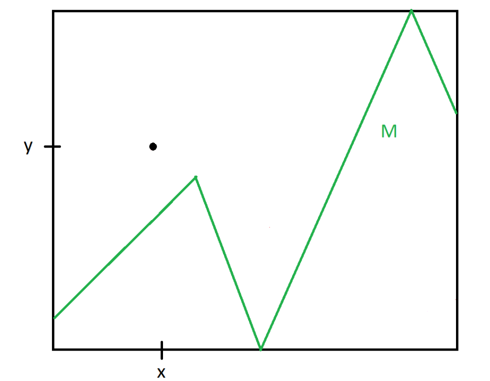
Let $t \in \mathbb{R}$. If the system $\mathcal{X} \times \mathcal{Y}_t$ has a dense orbit then it is minimal.

Proof.

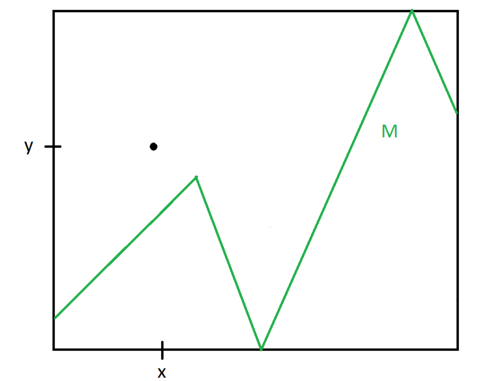
Let (x, y) be a point with a dense orbit. To show that $\mathcal{X} \times \mathcal{Y}_t$ is minimal, fix

- a nonempty, closed, invariant set $M \subseteq X \times Y$.

Proof.

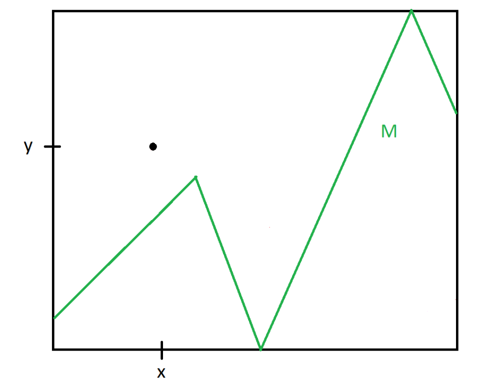


Proof.



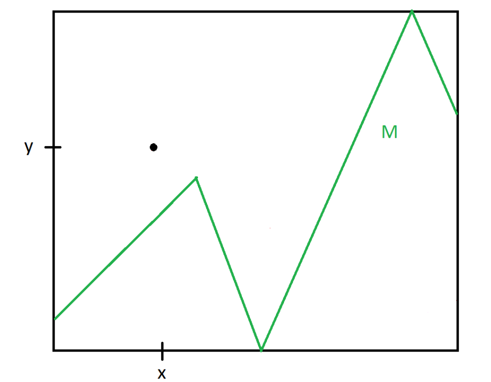
By compactness of Y, the projection $\mathrm{Pr}\colon X \times Y \to X$ is closed.

Proof.



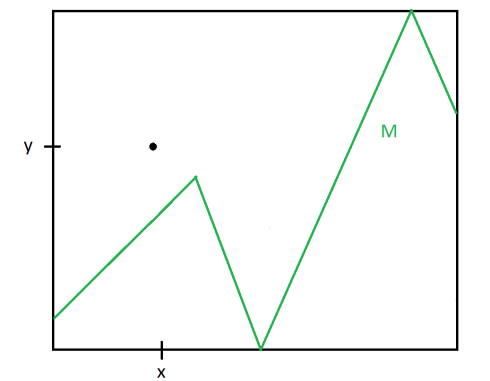
By compactness of Y, the projection $\mathrm{Pr}\colon X \times Y \to X$ is closed. Moreover, $\mathrm{Pr}\colon \mathcal{X} \times \mathcal{Y}_t \to \mathcal{X}$ is a homomorphism of dynamical systems.

Proof.



By compactness of Y, the projection $\mathrm{Pr}\colon X \times Y \to X$ is closed. Moreover, $\mathrm{Pr}\colon \mathcal{X} \times \mathcal{Y}_t \to \mathcal{X}$ is a homomorphism of dynamical systems. Consequently, by minimality of $\mathcal{X}$, M projects onto the whole of X.

Proof.



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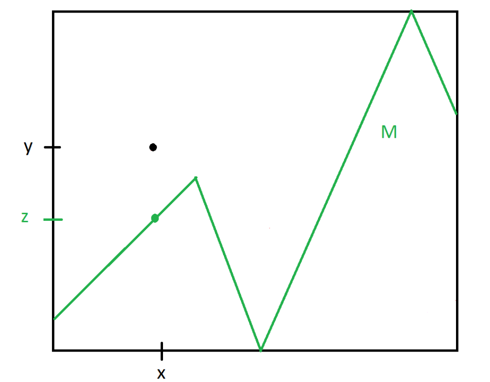
Proof.



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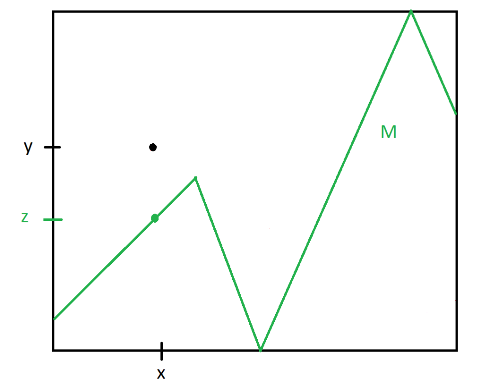
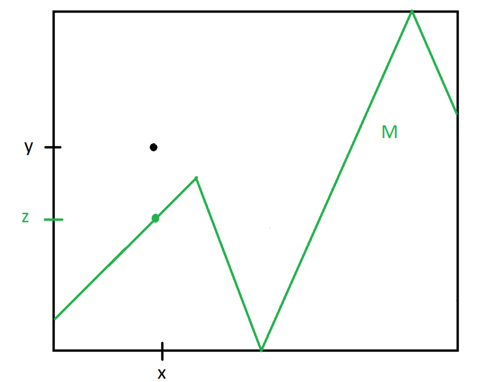


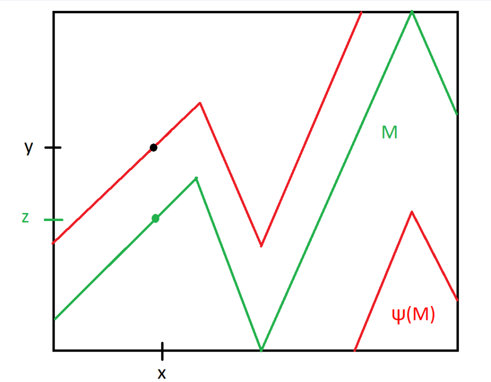
Now, $Z(\phi)$ acts transitively on Y, so $\psi(z) = y$ for some $\psi \in Z(\phi)$.

Proof.



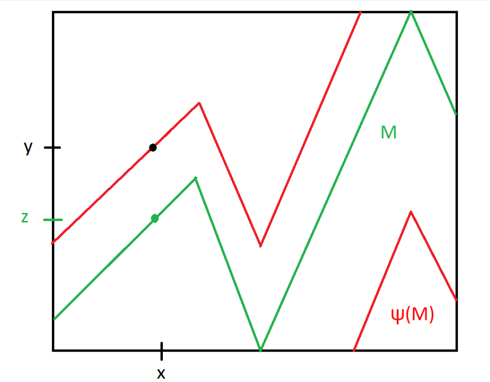
Now, $Z(\phi)$ acts transitively on Y, so $\psi(z) = y$ for some $\psi \in Z(\phi)$. If we consider ψ as a vertical homeomorphism on $X \times Y$ then it is an automorphism of $\mathcal{X} \times \mathcal{Y}_t$. □

Proof.



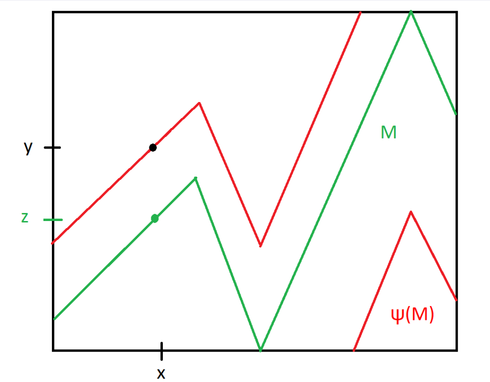
Consequently, $\psi(M)$ is a nonempty closed invariant set for $\mathcal{X} \times \mathcal{Y}_t$.

Proof.



Consequently, $\psi(M)$ is a nonempty closed invariant set for $\mathcal{X} \times \mathcal{Y}_t$. Since $(x, y) \in \psi(M)$ and (x, y) has dense orbit, $\psi(M) = X \times Y$, whence $M = X \times Y$.

Proof.



Consequently, $\psi(M)$ is a nonempty closed invariant set for $\mathcal{X} \times \mathcal{Y}_t$. Since $(x, y) \in \psi(M)$ and (x, y) has dense orbit, $\psi(M) = X \times Y$, whence $M = X \times Y$. Thus, $\mathcal{X} \times \mathcal{Y}_t$ is minimal. □

PM-spaces in examples of strange minimal spaces

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Theorem 18

Let X be a DST-space and let Y be a product-minimal path-connected space. Then

- $X \times Y$ *admits a minimal map,*
- $(X \times Y)^2$ *admits no minimal maps.*

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Theorem 19

Let X be a DST-space and $n \geq 2$ be an integer. Then

- $X \times \mathbb{T}^n$ *admits a minimal homeomorphism as well as a minimal non-invertible map,*
- $(X \times \mathbb{T}^n)^2$ *admits no minimal maps.*

(Product-)minimality of direct products — summary

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(IV) *The product of a minimal space and a product-minimal space may fail to be product-minimal.*

If X is a DST-space then X is minimal, the torus $\mathbb{T}^2$ is product-minimal, but $X \times \mathbb{T}^2$ is not product-minimal as mentioned in Theorem 19.

Two more results on minimal direct products

In 1979, Glasner and Weiss described a very powerful method for constructing minimal extensions of dynamical systems. One of their general results has the following (immediate) corollary.

Theorem 20 (Glasner, Weiss, 1979)

Let X, Y be compact minimal spaces and let X admit a minimal homeomorphism isotopic to the identity. Then the product $X \times Y$ is minimal.

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Theorem 20 (Glasner, Weiss, 1979)

Let X, Y be compact minimal spaces and let X admit a minimal homeomorphism isotopic to the identity. Then the product $X \times Y$ is minimal.

Theorem 21

Let Y be a product-minimal space and X be a non-degenerate compact metrizable space admitting a minimal homeomorphism isotopic to the identity. Then $X \times Y$ is product-minimal. (The same is true for the notion of homeo-product-minimality.)