On Hausdorff dimension of Julia sets of real Feigenbaum maps

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Discrete dynamical system associated to a polynomial

Let $f(z) = a_0 z^d + a_1 z^{d-1} + \ldots + a_{d-1} z + a_d$ be a polynomial, $d \in \mathbb{N}, a_i \in \mathbb{C}, z \in \mathbb{C}$. For $n \in \mathbb{N}$ denote by $f^n(z) = f(f(\ldots(f(z))\ldots))$ (iterated n times).

Discrete dynamical system associated to a function $f(z)$: given z_0 consider its orbit
$O(z_0) = \{z_0, z_1 = f(z_0), z_2 = f(z_1), z_3 = f(z_2), \ldots\}$, $z_n = f^n(z)$.

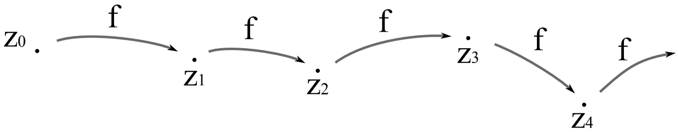


Figure: Orbit of a point.

The Julia set: informal definition

An important question is how orbits depend on the initial parameter z_0.

Natural dichotomy:
• regular behavior (slight change in the initial condition does not affect much the long-time behavior which can be therefore accurately predicted);
• chaotic behavior (arbitrary small variation of the initial condition may change unpredictably the long-time behavior).

Informally speaking, for a polynomial (or rational) function $f(z)$ the set of parameters z_0 producing chaotic behavior is called the *Julia set* J_f. Regular parameters constitute the Fatou set F_f.

The Julia set: formal definition

Filled Julia set $K_f = \{z \in \mathbb{C} : \{f^n(z)\}_{n\in\mathbb{N}} \text{ is bounded}\}$.
Julia set $J_f = \partial K_f$.

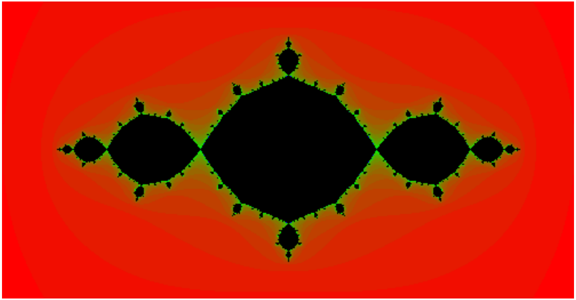


Figure: The Basilica Julia set, $f(z) = z^2 - 1$.

Julia set of a polynomial f

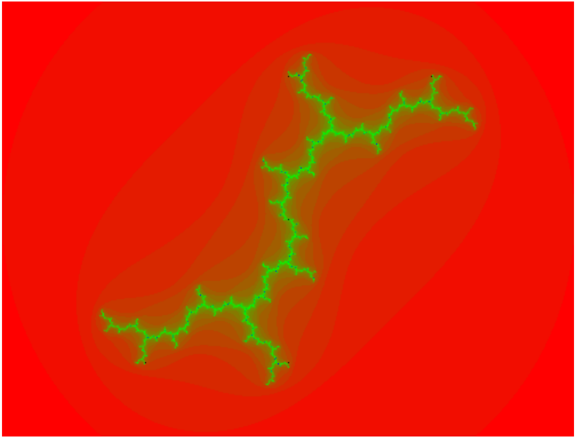


Figure: A dendrite Julia set, $f(z) = z^2 + i$.

Julia set of a polynomial f

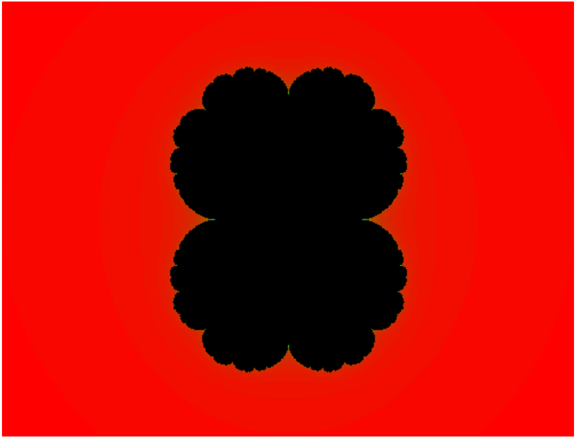


Figure: The cauliflower, $f(z) = z^2 + 0.25$.

Julia set of a polynomial f

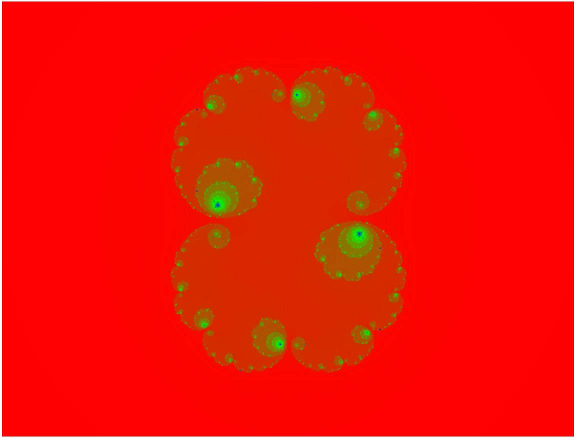
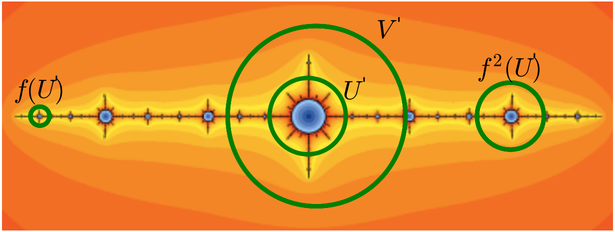


Figure: A perturbation of the cauliflower, $f(z) = z^2 + 0.26 + 0.001i$.

Renormalization

A *quadratic-like map* is a ramified covering $f : U \to V$ of degree 2, where $U \Subset V$ are topological disks in $\mathbb{C}$.
A quadratic-like map f is called *renormalizable with period p* if there exist domains $U' \Subset U$ for which $f^p : U' \to V' = f^p(U')$ is a quadratic-like map.



The map $f^n|_{U'}$ is called a *pre-renormalization of f*; the map $\mathcal{R}_p f := \Lambda \circ f^p|_{U'} \circ \Lambda^{-1}$, where Λ is an appropriate rescaling of U', is the *renormalization of f*.

Feigenbaum maps

Definition

A Feigenbaum map is an infinitely renormalizable quadratic-like map with bounded combinatorics and a priori bounds.

Avila-Lyubich '06, '07, '15: there exist Feigenbaum polynomials f_{c_n}, $c_n \to -2$, such that $\dim_{\mathrm{H}}(\mathcal{J}_{c_n}) \to 1$. There exists Feigenbaum polynomials f_c with $\mathrm{area}(\mathcal{J}_c) > 0$.
Assume a quadratic-like map $f : U \to V$ has $0 \in U$ as a critical point. We call f *real* if $f(\mathbb{R} \cap U) \subset \mathbb{R}$.
Open problems: Do there exist real Feigenbaum maps with positive area Julia sets (or at least of Hausdorff dimension 2)? Obtain bounds on the Hausdorff dimension of these Julia sets.

If f is renormalizable with period p we set $\Lambda(z) = z/\lambda,\ \lambda = f^p(0)$ so that $(\mathcal{R}_p(f))(0) = 1$.

Periodic points of renormalization

A quadratic-like map f is a periodic point of renormalization if

$$\mathcal{R}_p(f) = f \text{ for some } p, \text{ equivalently } f(z) = \frac{1}{\lambda} f^p(\lambda z).$$

By *Straigtening Theorem*, every quadratic-like map is *hybrid equivalent* (conjugated conformally on and quasi-conformally outside the Julia set) to a unique quadratic map $f_c(z) = z^2 + c$. For a real periodic point of renormalization f the corresponding map is a Feigenbaum map of *stationary combinatorics* (the relative positions of the iterates of critical orbit does not change under renormalization).

Periodic points of renormalization

Given an infinitely renormalizable map $f_c(z)$ with stationary combinatorics of period n the sequence of renormalizations $\mathcal{R}_n^k(f_c)$ converges to a periodic point F of renormalization: $\mathcal{R}_n(F) = F$.
Example 1: there is a unique period two infinitely renormalizable quadratic polynomical $f_{c_{\mathrm{Feig}}}(z) = z^2 + c_{\mathrm{Feig}}$,
$c_{\mathrm{Feig}} \approx -1.401155189092$ (discovered by Feigenbaum-Coullet-Tresser). One has $R_2^k(f_{c_{\mathrm{Feig}}}) \to F_{\mathrm{Feig}}$, where F_{Feig} is the fixed point of period two renormalization (also called the Feigenbaum map).

The Feigenbaum map

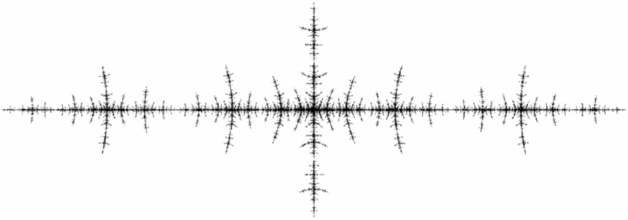


Figure: The Julia set of F_{Feig}

Theorem (D.-Sutherland)

The Julia set of F_{Feig} has Hausdorff dimension less than two (and hence its Lebesgue measure is zero).

In the first part of this talk: generalization of our approach to real periodic points of Feigenbaum renormalization and some numerical results.

Example 2: f_{c_3}

Example 2: real period 3 infinitely renormalizable quadratic polynomial is $f_{c_3}(z) = z^2 + c_3$ with $c_3 \approx -1.78644026$. One has $\mathcal{R}_3^k(f_{c_3}) \to F(z) \approx 1 - 1.87431z^2 + 0.09383z^4 - 0.00025z^6$, where $\mathcal{R}_3 F = F$.

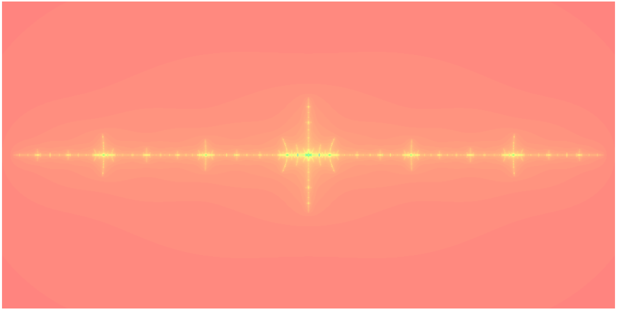


Figure: The Julia set of f_{c_3}.

Area and HD of quadratic Julia sets

Notations: $f_c(z) = z^2 + c$, $\mathcal{J}_c = \mathcal{J}_{f_c}$.

- Ruelle '82: $\dim_{\mathrm{H}}(\mathcal{J}_c)$ is real-analytic in c on hyperbolic components and outside of the Mandelbrot set.
- Shishikura '98: for a generic $c \in \partial\mathcal{M}$ one has $\dim_{\mathrm{H}}(\mathcal{J}_c) = 2$.
- McMullen '98: $\dim_{\mathrm{H}}(\mathcal{J}_c)$ is continuous on $(c_{\mathrm{Feig}}, \frac{1}{4}]$.
- McMullen '98, Jenkinson-Pollicott '02: effective algorithms for computing $\dim_{\mathrm{H}}$ of attractors of conformal expanding dynamical systems (*e.g.* hyperbolic Julia sets).
- Buff-Cheritat '12: there exist quadratic polynomials with positive area Julia sets *a*) having a Siegel fixed point, *b*) having a Cremer fixed point, *c*) infinitely satellite renormalizable.

Nice domains

Denote by f_n the n-th prerenormalization of f, by $\mathcal{J}_n$ the Julia set of f_n and by $\mathcal{O}(f)$ the critical orbit of f.

Avila and Lyubich constructed domains $U^n \subset V^n$ (called *nice domains*) for which

- $f_n(U^n) = V^n$;
- $U^n \supset \mathcal{J}_n \cap \mathcal{O}(f)$;
- $V^{n+1} \subset U^n$;
- $f^k(\partial V^n) \cap V^n = \emptyset$ for all n, k;
- $A^n = V^n \setminus U^n$ is "far" from $\mathcal{O}(f)$;
- $\mathrm{area}(A^n) \asymp \mathrm{area}(U^n) \asymp \mathrm{diam}(U^n)^2 \asymp \mathrm{diam}(V^n)^2$.

Escaping and returning sets

For each $n \in \mathbb{N}$, let X_n be the set of points in U^0 that land in V^n under some iterate of f:

$$X_n = \{z \in U^0 : f^k(z) \in V^n \text{ for some } n \geqslant 0\},$$

and let Y_n be the set of points in A^n that never return to V^n under iterates of f:

$$Y_n = \{z \in A^n : f^k(z) \notin V^n \text{ for all } n \geqslant 1\}.$$

Introduce the quantities

$$\eta_n = \frac{\mathrm{area}(X_n)}{\mathrm{area}(U^0)}, \quad \xi_n = \frac{\mathrm{area}(Y_n)}{\mathrm{area}(A^n)}.$$

Avila-Lyubich trichotomy

Theorem (Avila-Lyubich)

Let f be a periodic point of renormalization ($\mathcal{R}^p f = f$ for some p). Then exactly one of the following is true:

- η_n *converges to* 0 *exponentially fast,* $\inf \xi_n > 0$*, and* $\dim_{\mathrm{H}}(\mathcal{J}_f) < 2$ *(Lean case);*
- $\eta_n \asymp \xi_n \asymp \frac{1}{n}$ *and* $\dim_{\mathrm{H}}(\mathcal{J}_f) = 2$ *with* $\mathrm{area}(\mathcal{J}_f) = 0$ *(Balanced case);*
- $\inf \eta_n > 0$, ξ_n *converges to* 0 *exponentially fast, and* $\mathrm{area}(\mathcal{J}_f) > 0$ *(Black Hole case).*

The structure of real Feigenbaum periodic points

The Cvitanović-Feigenbaum equation:

$$\begin{cases} F(z) &= \frac{1}{\lambda} F^p(\lambda z), \\ F(0) &= 1, \\ F(z) &= H(z^2), \end{cases}$$

with $H'(0) \neq 0$.

Theorem (McMullen)

The map F has a maximal analytic extension to $\hat{F} : \hat{W} \to \mathbb{C}$, where $\hat{W} \supset \mathbb{R}$ is open, simply connected and dense in $\mathbb{C}$. All critical points of $\hat{F}$ are simple. The critical values of $\hat{F}$ are contained in real axis. Moreover, $\hat{F}$ is a ramified covering.

Using the above tile the plane by connected components of $\hat{F}^{-1}(\mathbb{H}_\pm)$.

We call *tiles* connected components of $\hat{F}^{-k}(\mathbb{H}_\pm)$. Notice that tiles are nested: for any two tiles P, Q one has $P \subset Q$ or $Q \subset P$ or $P \cap Q = \varnothing$. Tiles also are scaling invariant: if P is a tile, then so is λP.

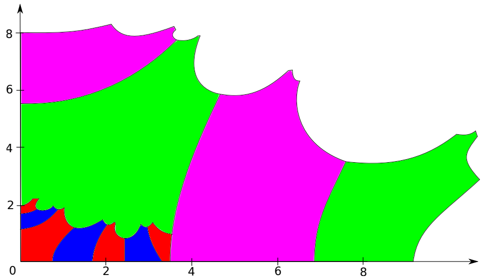


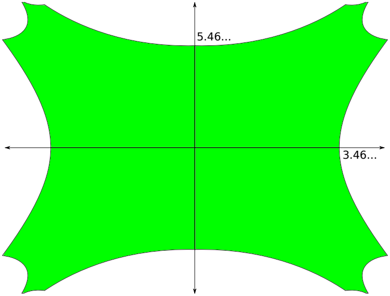
Figure: Some tiles for renormalization periodic point F of period 3 with $k = 1$ and $k = 2$.

Denote by $P_{\mathrm{I}}, P_{\mathrm{II}}, P_{\mathrm{III}}$ and P_{IV} the connected components of $\hat{F}^{-1}(\mathbb{H}_{\pm})$ containing 0 on the boundary. Set

$$W = \mathrm{Int}(\overline{P_{\mathrm{I}} \cup P_{\mathrm{II}} \cup P_{\mathrm{III}} \cup P_{\mathrm{IV}}}).$$

Then the restriction of $\hat{F}$ onto W is a quadratic-like map with the image of the form $\mathbb{C} \setminus ((-\infty, \alpha] \cup [\beta, \infty))$. We set $F = \hat{F}|_W$. For $n \in \mathbb{N}$ and any set A let $A^{(n)} = \lambda^n A$. Notice that $F_n = F^{2^n}|_{W^{(n)}}$ the n-th pre-renormalization of F.

Central tiles



The (new) returning and escaping sets

$$\tilde{X}_n = \{z \in W^{(1)} : F^k(z) \in W^{(n)} \text{ for some } k\},\ \tilde{\eta}_n = \frac{\text{area}(\tilde{X}_n)}{\text{area}(W^{(1)})}.$$

$$\widetilde{Y}_n = \{z \in W^{(n)} : F^k(z) \notin W^{(n)} \text{ for all } k \in \mathbb{N}\},\ \tilde{\xi}_n = \frac{\text{area}(\tilde{Y}_n)}{\text{area}(W^{(n)})}.$$

Using Avila-Lyubich trichotomy we obtain:

Proposition

$\dim_{\mathrm{H}}(\mathcal{J}_F) < 2$ *if and only if* $\tilde{\eta}_n \to 0$ *exponentially fast.*

Idea to prove $\tilde{\eta}_n \to 0$: construct recursive estimates of the form

$$\tilde{\eta}_{n+m} \leqslant C\tilde{\eta}_n\tilde{\eta}_m,$$

show that $C\tilde{\eta}_n < 1$ for some n.

Koebe space

Set $c_{\rm cl} = \min\{|F^l(0)| : 1 \leqslant l < p\}/|\lambda|$. Observe that $c_{\rm cl} \leqslant |F(0)|/|\lambda| = 1/|\lambda|$. Introduce the set

$$\mathbb{C}_{\rm cut} = \mathbb{C} \setminus ((-\infty, c_{\rm cl}] \cup [c_{\rm cl}, \infty)).$$

By Koebe Distortion Theorem there exists a constant C such that for any univalent function φ on $\mathbb{C}_{\rm cut}$ one has:

$$\frac{|\varphi'(x)|}{|\varphi'(y)|} \leqslant C, \quad \text{for all} \quad x, y \in W.$$

The main results

Theorem

For every $n, m \in \mathbb{N}$ one has

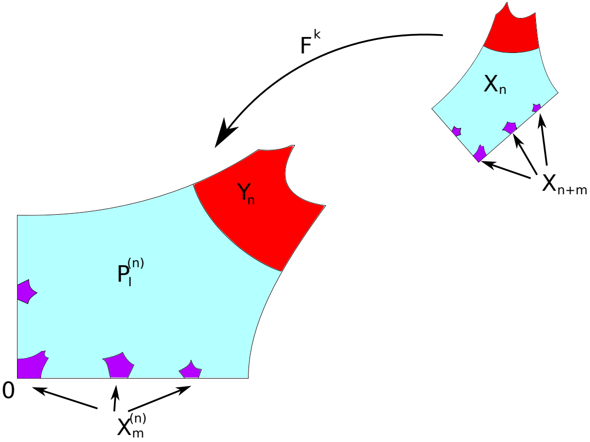
$$\tilde{\eta}_{n+m} \leqslant C^2 \tilde{\eta}_n \tilde{\eta}_{m+1} / \tilde{\xi}_n.$$

F of period p is uniquely determined by a permutation s on the set $\{0, 1, \ldots, p-1\}$ such that $F^j(0) < F^k(0)$ for $0 \leqslant j, k \leqslant p-1$ if and only if $s(j) < s(k)$. We obtain the following empirical results:

- For $F_{[1,0,2]}$: $\tilde{\eta}_3 < 0.0105$, $\xi_3 > 0.69$, $C^2 < 52$.
- For $F_{[1,0,3,2]}$: $\tilde{\eta}_3 < 0.01$, $\xi_3 > 0.3$, $C^2 < 26$.
- For $F_{[1,0,4,3,2]}$: $\tilde{\eta}_2 < 0.014$, $\xi_2 > 0.42$, $C^2 < 10.23$;
- For $F_{[2,0,4,3,1]}$: $\tilde{\eta}_2 < 0.014$, $\xi_2 > 0.43$, $C^2 < 22.2$;
- For $F_{[1,0,5,4,3,2]}$: $\tilde{\eta}_2 < 0.08$, $\xi_2 > 0.6$, $C^2 < 3.1$.

In each of the cases modulo numerical errors $C^2 \tilde{\eta}_n / \tilde{\xi}_n < 1$ and so $\dim_{\mathrm{H}}(J_F) < 2$.

Illustration to recursive estimates



McMullen's eigenvalue method: notations

The rest of the talk: application of the McMullen's algorithm to estimate from below the Hausdorff dimension of the Julia set of a real Feigenbaum quadratic map (ongoing joint work with Igors Gorbovickis).

Let $\mathcal{F}$ be a conformal dynamical system on $\mathbb{R}^n$ (*i.e.* a collection of conformal maps $f : U(f) \subset \mathbb{R}^n \to \mathbb{R}^n$). Let μ be an $\mathcal{F}$-invariant density of dimension δ (*i.e.* $\mu(f(E)) = \int\limits_E |f'(x)|^\delta d\mu$ whenever $f|E$ is injective, $E \subset U(f)$ is a Borel set and $f \in \mathcal{F}$). Let $\mathcal{P} = \{(P_i, f_i)\}$ be a finite expanding Markov partition for $(\mathcal{F}, \mu)$. The refinement $R(\mathcal{P})$ of $\mathcal{P}$ is the new Markov partition with pieces $P_{ij} = f_i^{-1}(P_j) \cap P_i$. Fix a sample point $x_i \in P_i$ for every $P_i \in \mathcal{P}$.

(1) For each i, j such that $\mu(f_i(P_i) \cap P_j) \neq 0$ let $y_{ij} \in P_i$ be such that $f_i(y_{ji}) = x_j$.
(2) Compute the transition matrix with the entries $T_{ij} = |f_i'(y_{ij})|^{-1}$ whenever $\mu(f_i(P_i) \cap P_j) \neq \varnothing$, and $T_{ij} = 0$ otherwise.
(3) Find $\alpha = \alpha(\mathcal{P}, \{x_i\})$ such that the spectral radius $\rho(T^\delta) = 1$, where T^δ means the matrix with the entries T_{ij}^δ.

Theorem (McMullen)

With the above conditions one has:

$$\alpha(R^n(\mathcal{P}) \to \delta \ \textit{exponentially fast.}$$

Theorem (McMullen)

For a hyperbolic polynomial $f(z)$ (i.e. expanding on its Julia set) there exists a unique invariant density μ of dimension $\delta = \dim_{\mathrm{H}}(J_f)$.

McMullen also showed:
One can define a partition $\mathcal{P}$ of a Julia set of a hyperbolic map f using external rays.
The eigenvalue algorithm applied to $\mathcal{P}$ computes $\dim_{\mathrm{H}}(J_f)$.

Hausdorff dimension of real Feigenbaum Julia sets

Let $f_c(z) = z^2 + c, c \in \mathbb{R}$, be a Feigenbaum map (*e.g.* has periodic combinatorics). Let $P_1 = \{z : \operatorname{Im} z > 0, |z| < 2\}$, $P_2 = \{z : \operatorname{Im} z < 0, |z| < 2\}$ and $\mathcal{P} = (P_1, P_2)$. Denote by $R^n(\mathcal{P})$ the collection of connected components of $f^{-n}(P_{1/2})$. Given $R^n(\mathcal{P}) = \{P_i^{(n)}\}$ introduce the matrix $T^{(n)}$ with entries

$$T_{ij}^{(n)} = (\sup\{|f_c'(y)| : y \in P_i^{(n)}, f_c(y) \in P_j^{(n)}\})^{-1},$$

if $f_c(P_i^{(n)}) \cap P_j^{(n)} \neq \varnothing$, and $T_{ij}^{(n)} = 0$ otherwise. Let α_n be such that $\rho((T^{(n)})_n^{\alpha}) = 1$.

Proposition (Dudko-Gorbovickis)

For each n one has $\alpha_n \leqslant \dim_{\mathrm{H}}(J_f)$.

Empiric estimates based on the above proposition suggest that the Hausdorff dimension of the Julia set of $f_{c_{\mathrm{Feig}}}$ is at least 1.41.

Other questions

- Given a real Fibonacci map $\mathrm{Fib}_d(z) = z^d + c_{\mathrm{Fib}}$ for which d it has a wild attractor? (Bruin-Keller-Nowicki-van Strien $\Rightarrow$ a wild attractor for sufficiently large d).
- Is there $d > 2$ such that Fib_d has the Julia set of positive area?
- Let $F_d(z)$ be the period 2 fixed point with critical point of degree $d > 2$. Is there d with $\dim_{\mathrm{H}}(J_{F_d}) = 2$ or $\mathrm{area}(J_{F_d}) > 0$? (Levin-Swiatek $\Rightarrow \dim_{\mathrm{H}}(J_{F_d}) \to 2$)
- Can one define appropriate tiling of the plane for complex periodic points of Feigenbaum renormalization?
- Construct sufficiently simple examples of Feigenbaum Julia sets of positive measure.
- Construct two-dimensional polynomial maps with Julia sets of positive measure.

Thank you!

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