Bernoulli disjointness
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First Part : Introduction

Flows (X, G)

A **flow** (X, G) consists of a pair X, a compact space, and G, a topological group, and a continuous homomorphism from G into the group $\mathrm{Homeo}(X)$ of all the self homeomorphisms of X. We will usually ignore this latter map and write gx for the image of $x \in X$ under the image of $g \in G$ in $\mathrm{Homeo}(X)$. By our assumption then the map $(g, x) \mapsto gx$ is continuous. A flow (X, G) is:

- **minimal** when Gx is dense in X for every $x \in X$. A point $x \in X$ is called minimal if its orbit closure $\overline{Gx}$ is minimal.

- **proximal** if for every $x, y \in X$ there is $z \in X$ and a net $g_i \in G$ such that $\lim g_i(x, y) = \lim(g_i x, g_i y) = (z, z)$.

- **strongly proximal** if for every probability measure $\mu \in P(X)$ there is $z \in X$ and a net $g_i \in G$ such that $\lim g_i \mu = \delta_z$.

Strong proximality implies proximality

If (X, G) is strongly proximal and $x, y \in X$, then we can form the measure $\mu = \frac{1}{2}(\delta_x + \delta_y)$. Now $\lim g_i \mu = \delta_z$ implies $\lim g_i(x, y) = (z, z)$. Thus strong proximality impies proximality.

• **incontractible** if for every $n \in \mathbb{N}$ the minimal points are dense in X^n.

• A closed invariant subset L of $X \times Y$ is a **joining** of the flows (X, G) and (Y, G) if $\pi_1(L) = X$ and $\pi_2(L) = Y$, where $\pi_i,\ i = 1, 2$ are the natural projections.

• Two flows (X, G) and (Y, G) are **disjoint** if $X \times Y$ is their unique joining. We denote this relation by $(X, G) \perp (Y, G)$.

• A surjective continuous map $\pi : (X, G) \to (Y, G)$ which intertwines the G actions is called a **factor map** or a **flow homomorphism**.

Universal flows

For many properties P of minimal flows there is a **universal** P-**flow**; i.e. a minimal flow (X, G) having property P which admits every other flow with this property as its factor, and moreover this universal P-flow is unique up to an automorphism. Among these properties we have (i) minimality, (ii) proximality, and (iii) strong proximality. The corresponding universal minimal flows are denoted by $M(G)$, $\Pi(G)$ and $\Pi_s(G)$ respectively.

Groups

- A topological group G is **amenable** if for every flow (X, G) the set $P_G(X)$ of invariant probability measures on X is nonempty. It can be shown that G is amenable iff $\Pi_s(G)$ is the trivial one point space. Solvable and compact groups are amenable. Non-abelian free groups are not.
- A topological group G is **strongly amenable** when $\Pi(G)$ is trivial.

• A topological group G is **maximally almost periodic** (maxap for short) if it has a continuous monomorphism into a compact group.

• The **FC center** of a group G is the collection of elements whose conjugacy class is finite. It is a characteristic subgroup of G.

• The group G is an **ICC group** if every element $e \neq g \in G$ has an infinite conjugacy class.

• The **FC radical** of a group G is the unique normal subgroup N of G such that G/N is ICC. It is obtained as an increasing union of successive (possibly transfinite) FC-centers.

A structure theorem

Theorem: Every group G either contains an infinite maxap normal subgroup, or it contains a normal subgroup $N \lhd G$ such that G/N is ICC.

Furstenberg's 1967 paper

In his famous 1967 paper "Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation" Furstenberg [Fur-67] has the following results:

- $\Omega = \{0,1\}^{\mathbb{Z}} \perp X$ for every minimal cascade (X, T).

- The subring $\mathcal{B}$ of Ω generated by the minimal functions is a proper subring of Ω.

He conjectured that, similarly, the sub-algebra $\mathcal{A}$ of $\ell^\infty(\mathbb{Z})$ generated by the minimal functions in $\ell^\infty(\mathbb{Z})$ is a proper sub-algebra of $\ell^\infty(\mathbb{Z})$.

Using these results he proved the following:

Theorem: For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the set $\{2^m 3^n \alpha : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Glasner-Weiss' paper from 1983

- A subset $A \subset \mathbb{Z}$ is an **interpolation set for a sub-algebra** $\mathcal{A} \subset \ell^\infty(\mathbb{Z})$ if for every $\omega \in \{0,1\}^A$ there is an element $f \in \mathcal{A}$ such that $f \restriction A = \omega$.
- A subset $A \subset \mathbb{Z}$ is an **small** if for every $N \in \mathbb{N}$ the set $\{n \in \mathbb{Z} : n + [1,N] \cap A = \emptyset\}$ is syndetic (i.e. has bounded gaps).

In [GW-83] the authors prove the following:

Theorem: The collection of interpolation sets for the sub-algebra $\mathcal{A} \subset \ell^\infty(\mathbb{Z})$, generated by the minimal functions, coincides with the ideal of small sets. In particular this shows that $\mathcal{A} \subsetneq \ell^\infty(\mathbb{Z})$.

Second Part : The main theorems

Bernoulli disjointness for the general infinite groups

We now list the main results in the recent work [GTWZ-21] :

Theorem A: For any infinite countable discrete group G one has:

(1) $\Omega = \{0,1\}^G \perp (X,G)$ for every minimal flow (X,G).

(2) The collection of interpolation sets for the sub-algebra $\mathcal{A} \subset \ell^\infty(G)$, generated by the minimal functions, coincides with the ideal of small sets. In particular this shows that $\mathcal{A} \subsetneq \ell^\infty(G)$.

Note: After our work had been circulated, Anton Bernshteyn [Ber-19] found a different proof of the fact that the Bernoulli flow is disjoint from minimal flows using the Lovász Local Lemma.

Corollary: Let G be an infinite discrete group and let $M(G)$ be its universal minimal flow. Then the canonical map from βG to the enveloping semigroup of $M(G)$ is not an isomorphism.

The question whether this map is an isomorphism was attributed to Robert Ellis and, for the general group G, was open for more than 50 years.

A continuum of pairwise disjoint minimal flows

For the integers group $\mathbb{Z}$, one has a continuum of minimal pairwise disjoint minimal flows, namely the collection of irrational rotations on the circle $(\mathbb{T}, R_\alpha)$, where α ranges over a Hamel basis for $\mathbb{R}$.

It follows that if $(X, \mathbb{Z})$ is a minimal metric flow then there is some α such that $(X, \mathbb{Z}) \perp (\mathbb{T}, \mathbb{R}_\alpha)$. In fact (X, T) can be not disjoint from at most a countable number of rotations.

Theorem B: Let G be an infinite countable group. Then

1. For every nontrivial minimal flow (X, G) there is a non trivial minimal flow (Y, G) with $(X, G) \perp (Y, G)$.
2. There is a collection of cardinality $\mathbf{c} = 2^{\aleph_0}$ of pairwise disjoint metric minimal flows.
3. $M(G) \cong \mathrm{Gleason}(\{0, 1\}^{\mathbf{c}})$.

The last result answers a question of Balcar and Błaszczyk who proved it for $\mathbb{Z}$.

Further results

Theorem C: A countable infinite ICC group G acts freely on its universal minimal proximal flow $\Pi(G)$.

This answers a question of Frisch, Tamuz, and Vahidi Ferdowsi, who have shown in [FTVF-19] that the action is effective..

Theorem D: Let G be a countable, infinite group. Then $\mathrm{Aut}(M(G), G)$ has cardinality $2^{\mathfrak{c}}$, the largest possible cardinality. In particular, $M(G)$ is not proximal.

Note that the flow $M(G)$ is a minimal left ideal of the right topological semigroup βG, and has the form $M(G) = J\mathfrak{G}$, where J is the set of idempotents in M and $\mathfrak{G}$ is a sub-group of M. Since right multiplication on M is continuous it is easy to see that $\mathfrak{G}$ can be identified with the group $\mathrm{Aut}(M(G), G)$.

To say that $\mathrm{Aut}(M(G), G)$ is trivial (i.e. consists of the identity element) is the same as saying that the flow $(M(G), G)$ is proximal; i.e. that $M(G) = \Pi(G)$.

Theorem E: Let G be an infinite countable group and let H be a maxap group, then there is a free minimal flow (X, G) with $H < \mathrm{Aut}(X, G)$.

Theorem E has been recently improved by Andy Zucker in [Zu-19]: there it is shown that for any two countable groups G and H with G infinite, there exists a minimal G-flow on the Cantor space such that H embeds in its automorphism group.

Third Part : The strategy of the proof of Theorem A

The separated covering property

Definition:

1. Let D be a finite subset of G. A subset $S \subset G$ is D-**spaced** if for every distinct $s_1, s_2 \in S$, we have $Ds_1 \cap Ds_2 = \emptyset$.
2. A minimal flow (X, G) has the **separated covering property (SCP)** if for every nonempty open set $U \subset X$ and any finite $D \subset G$, there is a D-spaced $S \subset G$ such that $S^{-1}U = X$.

Proposition 1: The following conditions on a minimal (X, G) are equivalent:

1. (X, G) has the SCP.
2. For every finite $D \subseteq G$, there is a D-spaced $S \subseteq G$ so that for every $x \in X$, $Sx \subseteq X$ is dense.
3. $(X, G) \perp \{0, 1\}^G$.

One free G-flow with SCP suffices

Proposition 2: If the group G admits an essentially free flow with the SCP then every minimal G-flow has the SCP.

Proposition 3: The Bernoulli shift is disjoint from all minimal, proximal flows. In particular, all proximal flows have the SCP.

The case of a normal maxap subgroup

Proposition 4: Suppose that G admits an infinite normal subgroup H which is maxap. Then G admits a free, minimal flow with the SCP.

The case of an ICC quotient

Proposition 5:(FTVF and GTWZ) Let G be a countable ICC group. Then there exists a metrizable, essentially free, minimal, proximal G-flow X.

The fact that there is an effective such action is proven in the remarkable work [FTVF-19]. They show that the groups for which $\Pi(G)$ is trivial are exactly the groups with no ICC quotients, and that for any group G

$$\ker(G \curvearrowright \Pi(G)) = \mathrm{FC-radical} = \mathrm{strongly\ amenable\ radical}.$$

They asked whether the action of an ICC group on $\Pi(G)$ is free and we show in our work that this is indeed the case.

Proving Theorem A

Collecting the results mentioned so far we can now complete the proof of Theorem A as follows:

The structure theorem reduces the proof to two cases.

• If G contains an infinite normal maxap group then we are done.

• Otherwise G contains a normal subgroup N, namely the FC-radical, such that G/N is ICC. If N is trivial G itself is ICC and again we are done.

If N is not trivial then also F, the **FC center** of G consisting of all elements of G with finite conjugacy classes, is nontrivial. Note that F is a characteristic subgroup of G.

When F is finite G/F is ICC and again a simple argument shows that G has SCP. If F is infinite, let Z be its center.

If Z is infinite, it is a normal maxap subgroup and we are done.

Finally, when Z is finite, the group $F' = F/Z$ is residually finite (this follows since every element of F has a finite conjugacy class in G). As such it is an infinite maxap normal subgroup of $G' = G/Z$ hence G' has SCP and again one deduces that also G has SCP.

The proof of Theorem A is complete.

Part Four : The space $\mathcal{S}(A^G)$, the closure of the strongly irreducible subshifts

Strongly and precisely irreducible subshifts

Definition: Let A be a finite set and G a countable group.

1. A subshift $X \subset A^G$ is said to be **strongly irreducible** if there exists a finite set $D \subset G$ such that for every two finite sets $E_1, E_2 \subset G$ which are D-spaced and for every $x_1, x_2 \in X$ there is $x \in X$ with

$$x \restriction E_1 = x_1 \restriction E_1 \quad \text{and} \quad x \restriction E_2 = x_2 \restriction E_2.$$

2. A subshift $X \subset (A^{\mathbb{N}})^G$ is said to be **precisely irreducible** if there exists a finite set $D \subset G$ such that for every two finite sets $E_1, E_2 \subset G$ which are D-spaced and for every $x_1, x_2 \in X$ there is $x \in X$ with

$$x \restriction E_1 = x_1 \restriction E_1 \quad \text{and} \quad x \restriction E_2 = x_2 \restriction E_2.$$

Residual properties of $\mathcal{S}((A^{\mathbb{N}})^G)$

Let $\mathcal{S}((A^{\mathbb{N}})^G)$ denote the closure of the collection of the precisely irreducible subshifts.

Theorem: The following are dense G_δ subsets of $\mathcal{S}((A^{\mathbb{N}})^G)$:

1. $\{Z : Z \text{ is minimal}\}$
2. $\{Z : Z \text{ is essentially free}\}$
3. Given a minimal G-flow X, $\{Z : Z \perp X\}$.

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