Entropy in the context of aperiodic order

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Introduction

Definition (Word counting entropy)

For $a = (a_k)_{k \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ we define the *(Word counting) entropy* as

$$h_{\mathbf{word}}(a) := \lim_{n \to \infty} \frac{\log |W_n(a)|}{n},$$

where $W_n(a) := \{(a_l)_{l=k+1}^{k+n};\ k \in \mathbb{Z}\}$.

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Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^n$ *in* $\{0,1\}$*:*

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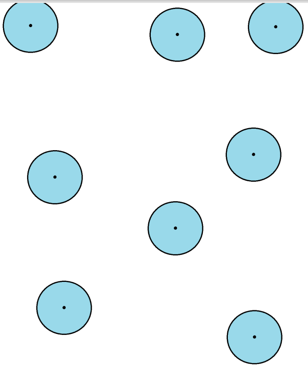
$\omega \subseteq G$ *uniformly discrete*, whenever there is an open neighbourhood $V \subseteq G$ such that $\{V + x;\, x \in \omega\}$ is a disjoint family.



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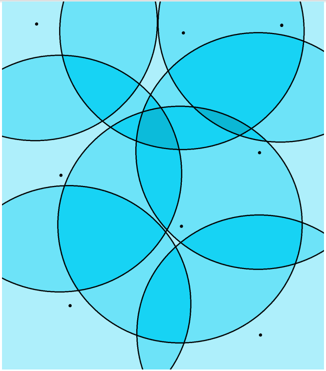

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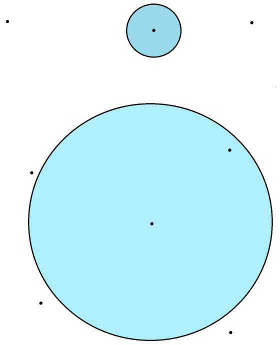
$\omega \subseteq G$ *Delone*, whenever ω uniformly discrete and relatively dense.



Patches of a Delone set

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Let $\omega \subseteq G$ be a Delone set. For compact subsets $A \subseteq G$ we define the *set of A-patches* as

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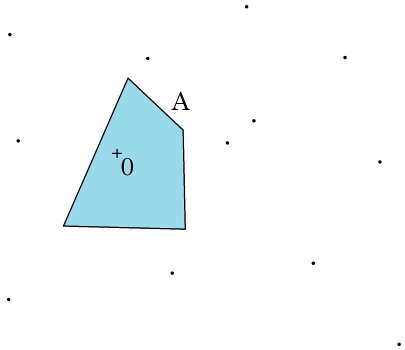

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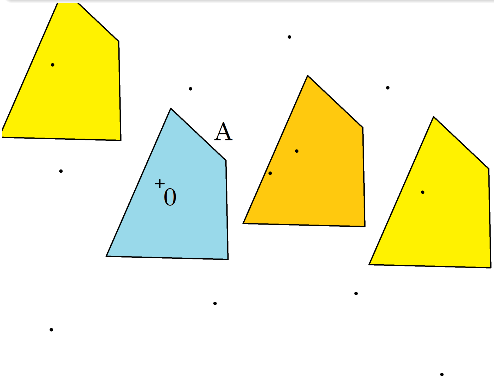

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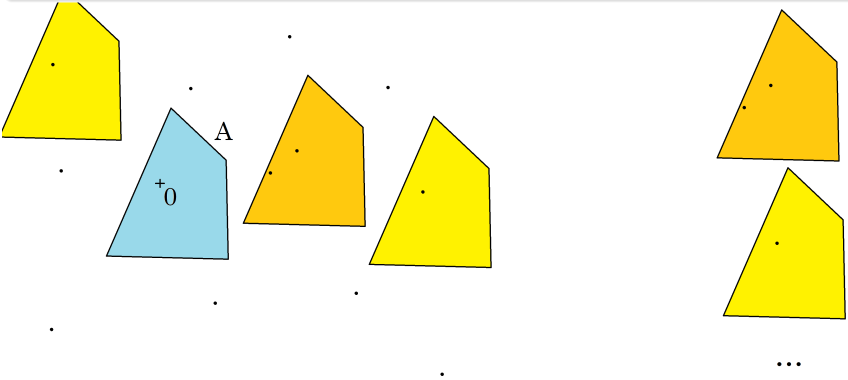

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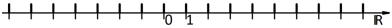
$\mathbb{Z} \subseteq \mathbb{R}$ is a FLC Delone set.



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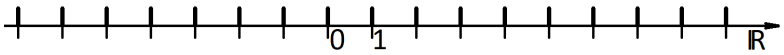

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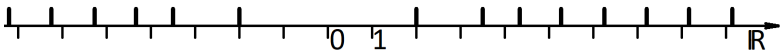
$\{n + 1/n;\ n \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{R}$ is a Delone set but not FLC.



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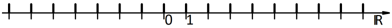

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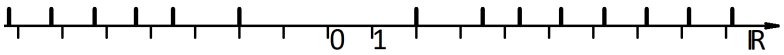

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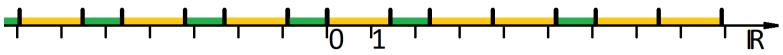

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Definition (Van Hove boundary)

For $K, A \subseteq G$ compact:

$\partial_K A := (K + A) \cap (K + \overline{G \setminus A})$,

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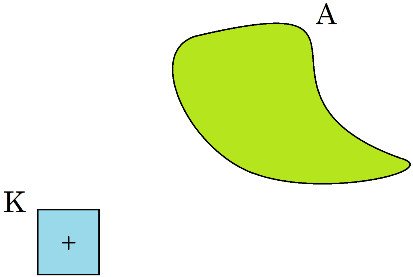

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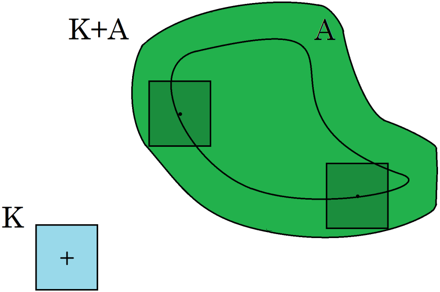

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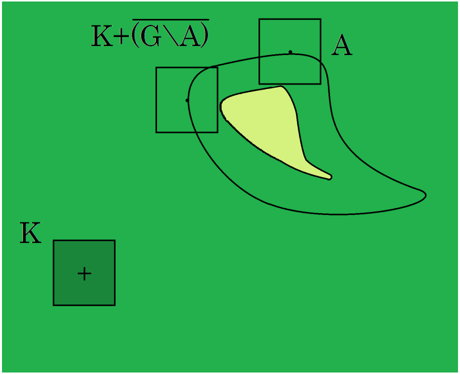

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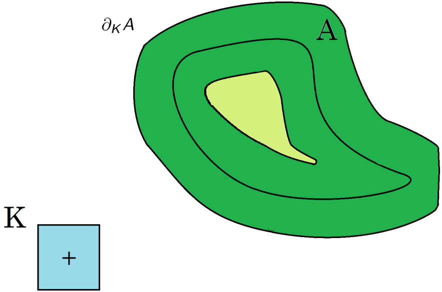

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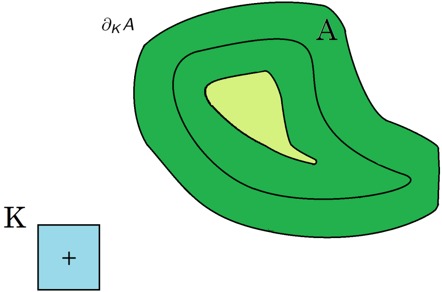

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Remark

Any σ-cpt. LCA group allows for a metric s.t. $(B_n(0))_{n\in\mathbb{N}}$ is a van Hove sequence.

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Choose metric d such that $(B_n(0))_{n\in\mathbb{N}}$ is a van Hove sequence in G.

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Theorem (J. Lagarias, 1999)

$h_{pat}(\omega) < \infty$ for FLC Delone sets in $\mathbb{R}^d$.

Delone action and topological entropy

Definition (Delone action π_ω)

For ω Delone set in G:

$$\pi_\omega \colon G \times X_\omega \to X_\omega,$$
$$\pi_\omega(g, \xi) := \xi + g$$

$$X_\omega := \overline{\{\omega + g;\, g \in G\}}^{\mathcal{A}(G)},$$
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Consider a van Hove sequence $(A_n)_{n\in\mathbb{N}}$ and a Delone set ξ in G.

Delone action and topological entropy

Definition (Delone action π_ω)

For ω Delone set in G:

$\pi_\omega \colon G \times X_\omega \to X_\omega$,
$\pi_\omega(g, \xi) := \xi + g$

$X_\omega := \overline{\{\omega + g;\, g \in G\}}^{\mathcal{A}(G)}$,
$\mathcal{A}(G) := \{A \subseteq G \text{ closed}\}$
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Theorem (T.H. and F. M. Schneider)

Notion is independent from the choice of ξ and $(A_n)_{n\in\mathbb{N}}$

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Theorem (M. Baake, D. Lenz, C. Richard, T. H.)

If G is compactly generated then

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