Minimal sets on continua with dense free intervals

(work in progress)

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Dynamical system

A **dynamical system** is a pair (X, f) where X is a compact metrizable space and $f\colon X \to X$ is a continuous map.

Minimal set of a system

A set $M \subseteq X$ is a **minimal set** of (X, f) if M is a nonempty, closed, f-invariant and there is no proper subset of M having these three properties.

Minimal set on a space

If $M \subseteq X$ is a minimal set of some dynamical system on a space X, we call M a **minimal set on the space** X.

Free interval

A **free interval** in X is an open set homeomorphic to $(0, 1)$.

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Continuum

A topological space X is a **continuum** if X is a nonempty compact connected metrizable space.

Locally connectedness

A topological space X is **locally connected at** $x \in X$ if for every neighborhood U of x there exists a connected neighborhood $V \subseteq U$ of x.
X is **locally connected** if it is locally connected at each of its points.

Notation

For a metrizable (not necessarily compact) space Y, the symbol $\mathcal{M}(Y)$ denotes the system of all minimal sets on Y.

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Theorem [Birkhoff]

Any dynamical system has a minimal set.

The complete characterization of topological structure of minimal sets is known on:

- zero-dimensional compact metrizable spaces:
  - finite or Cantor sets
- the unit interval:
  - finite or Cantor sets
- the circle:
  - finite or Cantor sets or the entire circle
- graphs:
  - finite or Cantor sets or unions of finitely many pairwise disjoint circles
- ...

Theorem [Balibrea, Downarowicz, Hric, Snoha, Špitalský (2009)]

Let X be a local dendrite (i.e., a locally connected continuum with finitely many circles).
Then M is a minimal set on X if and only if one of the following conditions holds:

- M is a finite set;
- M is a cantoroid (i.e., a compact metrizable space without isolated points where degenerate components are dense);
- M is a union of finitely many pairwise disjoint circles.

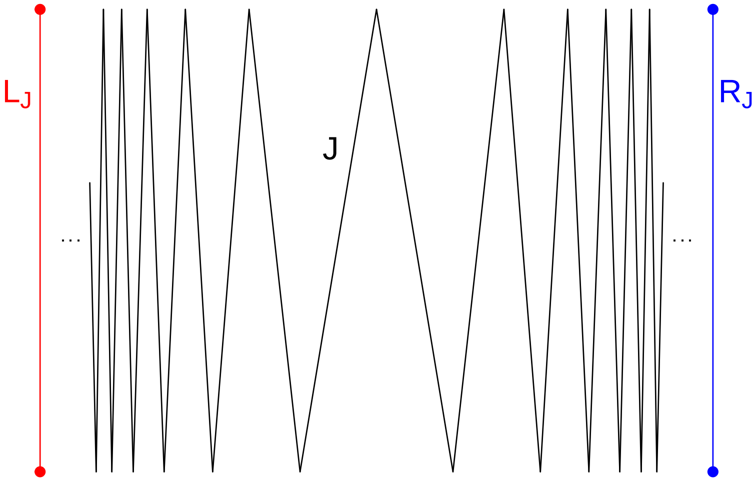
Continua with dense free intervals

$$X = L_J \cup J \cup R_J$$

- X is a continuum,
- J is a free interval **dense** in X,
- L_J, R_J are nowhere dense locally connected continua disjoint with J

- X: a compactification of (the real line) J
- L_J, R_J: remainders

Continua with dense free intervals



Continua with dense free intervals

Theorem [Martínez-de-la-Vega, Minc (2014)]

For each nondegenerate continuum P there is uncountably many topologically distinct compactifications of $[1, \infty)$ each with P as the remainder.

Corollary

For every nondegenerate continua L, R there are uncountably many topologically distinct spaces $X = L_J \cup J \cup R_J$ such that

- L_J is homeomorphic to L,
- R_J is homeomorphic to R.

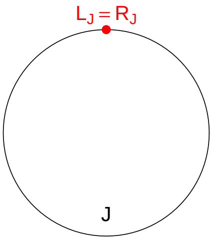
Continua with dense free intervals

Up to the symmetry, there are three possibilities for $X = L_J \cup J \cup R_J$:

0. both L_J, R_J are singletons; X is path connected,
1. R_J is a singleton, L_J is nondegenerate
   - if $R_J \subseteq L_J$, then X is path connected
   - if $L_J \cap R_J = \emptyset$, then the path components are L_J and $J \cup R_J$
2. both L_J, R_J are nondegenerate
   - if $L_J \cap R_J \neq \emptyset$, then the path components are $J, L_J \cup R_J$
   - if $L_J \cap R_J = \emptyset$, then the path components are J, L_J, R_J

(0a) L_J, R_J are singletons and $L_J = R_J$

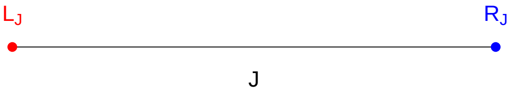
X is homeomorphic to a circle.



$$\mathcal{M}(X) = \{M \subseteq X\colon M \text{ is a finite set or a Cantor set or } X\}$$

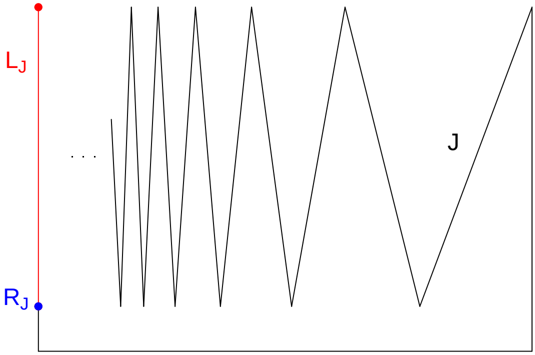
(0b) L_J, R_J are singletons and $L_J \neq R_J$

X is homeomorphic to a compact interval.



$$\mathcal{M}(X) = \{M \subseteq X \colon M \text{ is a finite set or a Cantor set}\}$$

(1a) L_J is a nondegenerate continuum, R_J is a singleton and $R_J \subset L_J$



Warsaw circle

(1a) L_J is a nondegenerate continuum, R_J is a singleton and $R_J \subset L_J$

Theorem

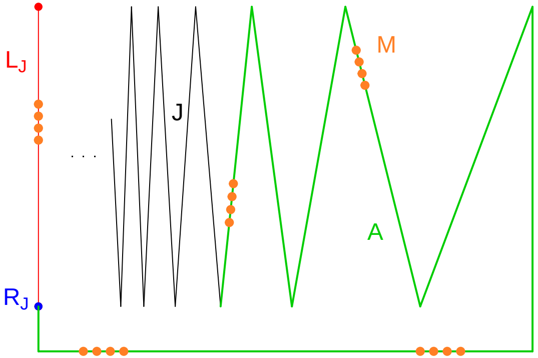
$$\mathcal{M}(X) = \bigcup \{\mathcal{M}(L_J \cup A) : A \text{ is an arc, } R_J \subseteq A \subseteq R_J \cup J\}$$

Corollary

Moreover, if L_J is a local dendrite, then $M \subseteq X$ is a minimal set on X if and only if one of the following conditions holds:

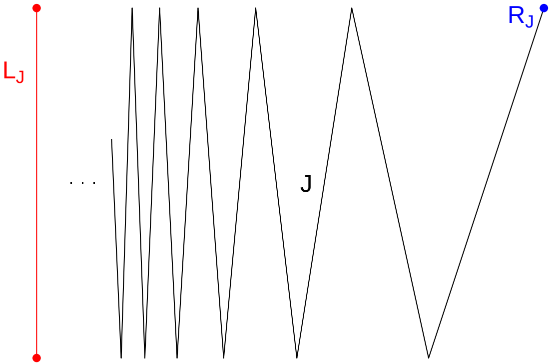
- M is a finite set;
- M is a union of finitely many pairwise disjoint circles;
- M is a cantoroid and $M \subseteq L_J \cup A$ for an arc $A \subseteq R_J \cup J$ containing R_J.

(1a) Example: X = Warsaw circle



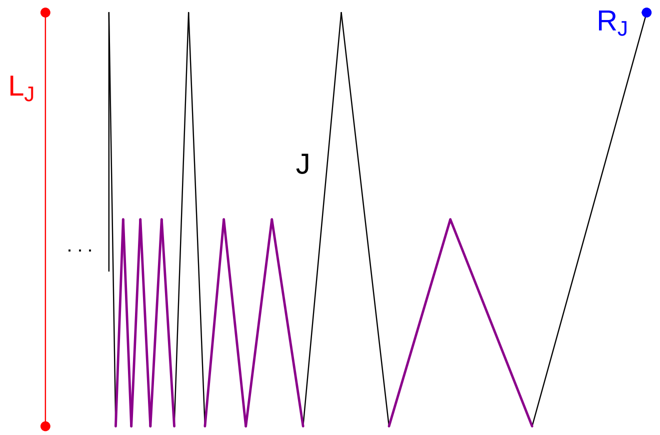
$$M \in \mathcal{M}(X) \iff M \text{ is a finite or Cantor set in } L_J \cup A$$

(1b) L_J is a nondegenerate continuum, R_J is a singleton and $R_J \cap L_J = \emptyset$



Topologist's sine curve

(1b) L_J is a nondegenerate continuum, R_J is a singleton and $R_J \cap L_J = \emptyset$



(1b) L_J is a nondegenerate continuum, R_J is a singleton and $R_J \cap L_J = \emptyset$

Theorem

$$\mathcal{M}(X) = \mathcal{M}(L_J) \sqcup \mathcal{M}(J \cup R_J)$$

and $\mathcal{M}(J \cup R_J)$ is the system of all finite and Cantor subsets of $J \cup R_J$.

(1b) L_J is a nondegenerate continuum, R_J is a singleton and $R_J \cap L_J = \emptyset$

Corollary

Moreover, if L_J is a local dendrite, then $M \subseteq X$ is a minimal set on X if and only if exactly one of the following conditions holds:

- $M \subseteq L_J$ such that M is either a finite set or a cantoroid or a union of finitely many pairwise disjoint circles;
- $M \subseteq J \cup R_J$ such that M is either a finite set or a Cantor set.

Example: X = Topologist's sine curve

$$M \in \mathcal{M}(X) \iff \begin{cases} M \text{ is a finite or Cantor set in } L_J, \\ M \text{ is a finite or Cantor set in } J \cup R_J. \end{cases}$$

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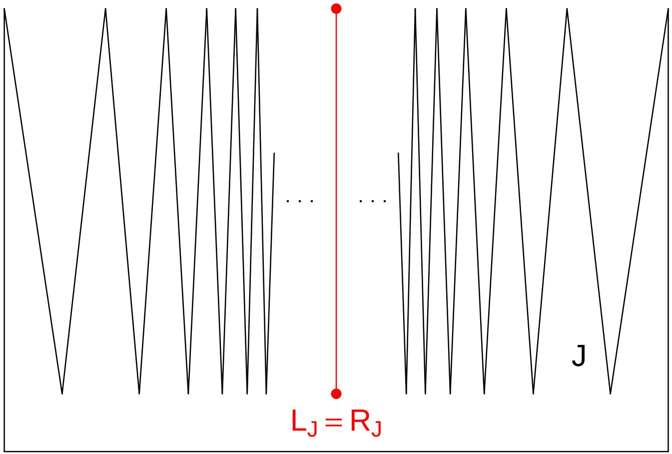
(2a) L_J, R_J are nondegenerate continua and $R_J \cap L_J \neq \emptyset$



(2a) L_J, R_J are nondegenerate continua and $R_J \cap L_J \neq \emptyset$

Theorem

$$\mathcal{M}(X) = \mathcal{M}(L_J \cup R_J) \sqcup \mathcal{M}(J)$$

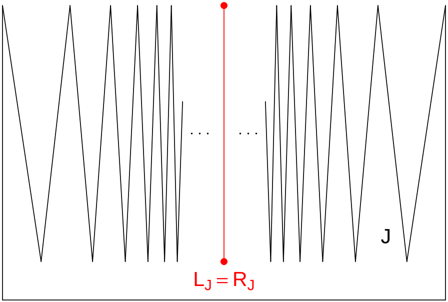
and $\mathcal{M}(J)$ is the system of all finite and Cantor subsets of J.

Corollary

Moreover, if L_J, R_J are local dendrites, then $M \subseteq X$ is a minimal set on X if and only if exactly one of the following conditions holds:

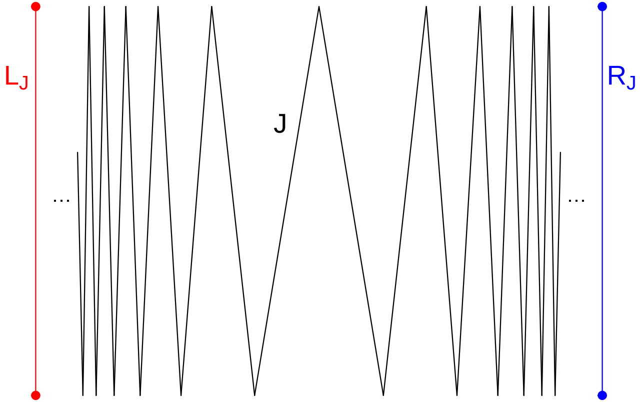
1. $M \subseteq L_J \cup R_J$ such that M is either a finite set or a cantoroid or a union of finitely many pairwise disjoint circles;
2. $M \subseteq J$ such that M is either a finite set or a Cantor set.

(2a) Example



$$M \in \mathcal{M}(X) \iff \begin{cases} M \text{ is a finite or Cantor set in } L_J \cup R_J, \\ M \text{ is a finite or Cantor set in } J. \end{cases}$$

(2b) L_J, R_J are nondegenerate continua and $R_J \cap L_J = \emptyset$



Double topologist's sine curve

(2b) L_J, R_J are nondegenerate continua and $R_J \cap L_J = \emptyset$

Notation

For metrizable (not necessarily compact) disjoint spaces Y and Z, the symbol $\mathcal{M}^*(Y;Z)$ denotes the system of all minimal sets M on $Y \cup Z$ such that the cardinality of $M \cap Y$ is equal to the cardinality of $M \cap Z$,

$$\operatorname{card}(M \cap Y) = \operatorname{card}(M \cap Z).$$

Theorem

1. $\mathcal{M}(X) \subseteq \mathcal{M}(J) \sqcup \mathcal{M}(L_J) \sqcup \mathcal{M}(R_J) \sqcup \mathcal{M}^*(L_J; R_J)$,
2. $\mathcal{M}(X) \supseteq \mathcal{M}(J) \sqcup \mathcal{M}(L_J) \sqcup \mathcal{M}(R_J)$

and $\mathcal{M}(J)$ is the system of all finite and Cantor subsets of J.

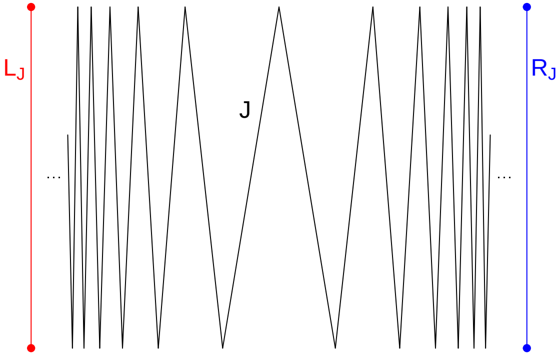
(2b) L_J, R_J are nondegenerate continua and $R_J \cap L_J = \emptyset$

Corollary

Moreover, if L_J, R_J are local dendrites, then:

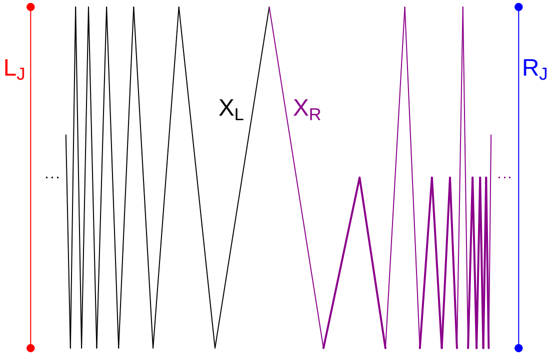
- if $M \subseteq X$ is a minimal set on X, then exactly one of the following conditions holds:
  1. $M \subseteq J$ and M is either a finite set or a Cantor set;
  2. either $M \subseteq L_J$ or $M \subseteq R_J$ such that M is either a finite set or a cantoroid or a union of finitely many pairwise disjoint circles;
  3. $M \subseteq L_J \cup R_J$, the sets $M \cap L_J$, $M \cap R_J$ have the same cardinalities and M is either a finite set or a cantoroid or a union of finitely many pairwise disjoint circles.
- if any of the conditions (1), (2) holds, then M is a minimal set on X.

(2b) Example 1: X = Double topologist's sine curve



$$\mathcal{M}(X) = \mathcal{M}(J) \sqcup \mathcal{M}(L_J) \sqcup \mathcal{M}(R_J) \sqcup \mathcal{M}^*(L_J; R_J)$$

(2b) Example 2



$$\mathcal{M}(X) = \mathcal{M}(J) \sqcup \mathcal{M}(L_J) \sqcup \mathcal{M}(R_J)$$

(2b) Question

Is there a space $X = L_J \cup J \cup R_J$ (with L_J and R_J disjoint nondegenerate) such that

$$\mathcal{M}(X) \cap \mathcal{M}^*(L_J; R_J) \neq \emptyset,$$

but

$$\mathcal{M}(X) \not\supseteq \mathcal{M}^*(L_J; R_J)$$

?

Thanks for your attention!