Specification property for step skew products

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1. Step skew products

$T_1, T_2 \colon I \to I$ continuous ($I = [0,1]$)

$x_0 \in I$ ... we decide, by tossing a coin, whether we take $x_1 = T_1(x_0)$ or $x_1 = T_2(x_0)$

... after $(n-1)$ steps arriving at x_{n-1} we again decide, tossing a coin, whether we take $x_n = T_1(x_{n-1})$ or $x_n = T_2(x_{n-1})$

$(x_n)_{n=0}^{\infty}$ ... trajectory of x_0 in the **nonautonomous system** given by the sequence of maps determined by the coin tossing, each of the maps being either T_1 or T_2

Since any choice of $\omega = \omega_0\omega_1\omega_2 \cdots \in \Sigma_2^+ = \{1,2\}^{\mathbb{Z}_+}$ yields a nonautonomous system given by the sequence of maps $T_{\omega_0}, T_{\omega_1}, T_{\omega_2}, \dots$, all such nonautonomous systems are in a sense present in the **skew product**

$$(\omega, x) \mapsto (S(\omega), T_{\omega_0}(x))$$

where S is the shift transformation $\Sigma_2^+ \to \Sigma_2^+$, $(S\omega)_n = \omega_{n+1}$.

1. Step skew products

Straightforward generalization:

- $T_1, T_2, \ldots, T_n \colon I \to I$ continuous (instead of T_1, T_2)
- $\Sigma_n^+ = \{1, 2, \ldots, n\}^{\mathbb{Z}_+}$ (instead of Σ_2^+)
- subshift $B \subseteq \Sigma_n^+$ (instead of full shift Σ_n^+)

The step skew product $F \colon B \times I \to B \times I$ is defined by

$$F(\omega, x) = (S(\omega), T_\omega(x))$$

where S is the shift transformation on $B \subseteq \Sigma_n^+$ and the continuous fibre map T_ω depends only on the beginning coordinate of ω, i.e.

$$\omega = \omega_0\omega_1\omega_2 \cdots \in B \Longrightarrow T_\omega = T_{\omega_0} \in \{T_1, T_2, \ldots, T_n\}$$

Clearly, F is continuous ($B \times I$ is endowed with the max metric).

1. Step skew products

Dynamics of step skew products is usually studied under additional assumptions on the fibre maps. For instance, the fibre maps are:

- interval C^1-maps (e.g. Kudryashov 2010)
- interval C^2-diffeomorphisms fixing the endpoints of I (e.g. Ilyashenko 2010)
- interval diffeomorphisms mapping I strictly inside itself (e.g. Kleptsyn and Volk 2014)
- interval C^1-diffeomorphisms onto their images (e.g. Gelfert and Oliveira 2020)
- interval maps T_1, T_2 fixing the endpoints of I, T_1 above and T_2 below the diagonal in the interior of I (e.g. piecewise linear homeomorphisms in Alseda and Misiurewicz 2014, diffeomorphisms in Gharaei and Homburg 2017)
- circle C^1-diffeomorphisms (e.g. Díaz, Gelfert and Rams, e.g. 2019)
- circle rotations (e.g. Falcó 1998, Mazur and Oprocha 2015)

2. Specification property

Dyn. system (Y, R) ... Y compact metric space with metric d,
$R\colon Y \to Y$ continuous

(Y, R) has the specification property, if

$(\forall \varepsilon > 0)\ (\exists M = M(\varepsilon))$:

$(\forall k \geq 2)\ (\forall (y_1, n_1), (y_2, n_2), \ldots, (y_k, n_k) \in Y \times \mathbb{N})\ (\exists u \in Y)$:

$R^{r_k}(u) = u$ and

$d(R^i(y_j), R^{r_{j-1}+i}(u)) \leq \varepsilon \text{ for } 0 \leq i < n_j \text{ and } 1 \leq j \leq k$

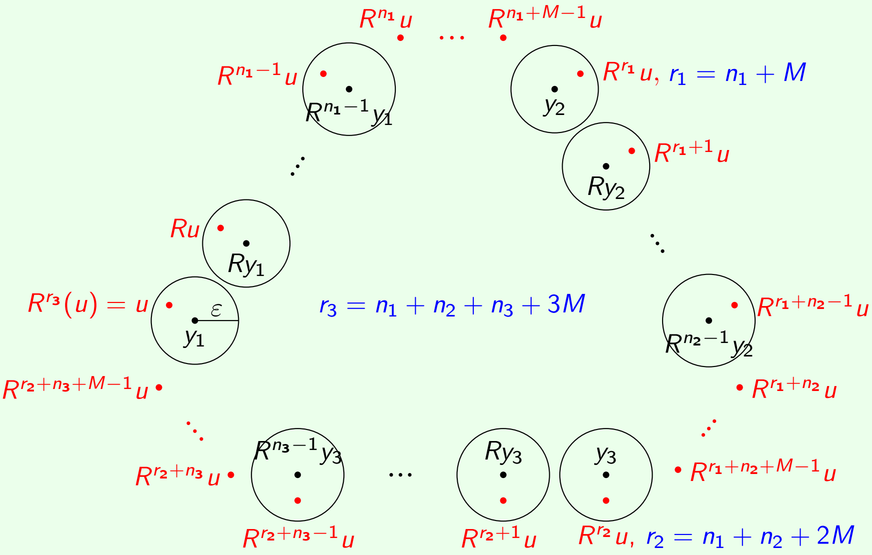
where $r_0 = 0$ and $r_j = n_1 + n_2 + \ldots + n_j + jM$ for $1 \leq j \leq k$.

We call M the gap length for the given ε.

Thus, $M = M(\varepsilon)$ is such that for every finite family of orbit segments, if all the gap lengths are prescribed to be equal to M, an ε-tracing periodic point u does exist.

(equivalent with the definition in which all the gap lengths are prescribed and greater than or equal to M)

2. Specification property



2. Specification property

The specification property was introduced by Bowen. Sometimes it is called *periodic* specification property (to distinguish from some variants introduced later).

specification property $\Longrightarrow$ dense subset of periodic points
topological mixing

On the interval (Blokh):

specification property $\Longleftrightarrow$ topological mixing

3. Finite nonautonomous systems and non-shrinking of intervals

$T: [0,1] \to [0,1]$

... piecewise monotone, if $\exists$ finite partition $\mathcal{P}$ of $[0,1]$ into intervals, such that $T|_P$ is monotone for every $P \in \mathcal{P}$

... expanding, if $\exists \alpha > 1$ such that $|T(x) - T(y)| \geq \alpha |x - y|$ holds for all x, y which are in the same element of $\mathcal{P}$

($\alpha =$ the expansion rate)

Nonautonomous system

... sequence $(f_i)_{i=0}^{\infty}$ of maps $[0,1] \to [0,1]$

(*finite*, if only finitely many different maps occur)

$f_j^i := f_{j+i-1} \circ \cdots \circ f_{j+1} \circ f_j$, in particular $f_0^i = f_{i-1} \circ \cdots \circ f_1 \circ f_0$

$|J| =$ length of an interval J

3. Finite nonautonomous systems and non-shrinking of intervals

Theorem 1

$T_1, \ldots, T_n \colon [0,1] \to [0,1]$ *expanding, piecewise monotone, cont.*

Then

$\forall \varepsilon > 0\ \exists \gamma > 0\ \forall$ *nonaut. system* $(f_i)_{i=0}^{\infty}$ *with* $f_i \in \{T_1, \ldots, T_n\}\ \forall i$:

$$U \text{ interval}, |U| \geq \varepsilon \implies \inf_{i \geq 0} |f_0^i(U)| \geq \gamma.$$

3. Finite nonautonomous systems and non-shrinking of intervals

Strategy of proof

1. Fix $\boxed{\varepsilon} > 0$ (we need $\gamma > 0$ s.t. if $|U| \geq \varepsilon$ then its trajectory consists of intervals whose lengths are $\geq \gamma$).
2. $\alpha := \min\{\text{exp. rates of } T_1, \ldots, T_n\} > 1 \Rightarrow \exists m: \alpha^m > 2$
3. Each T_i has finitely many critical points. Therefore: $\triangle = (H_1, \ldots, H_m) \in \{T_1, \ldots, T_n\}^m \Rightarrow \exists \beta_\triangle > 0 \; \forall \; |U| \leq \beta_\triangle$ which has a critical point of H_1 as endpoint, we have:

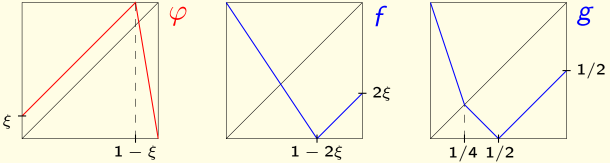
   U .... has no crit. pt. of H_1 in its interior
   $H_1(U)$ .... has no crit. pt. of H_2 in its interior
   $H_2(H_1(U))$ .... has no crit. pt. of H_3 in its interior
   ...
   $(H_{m-1} \circ \cdots \circ H_1)(U)$ ... has no crit. pt. of H_m in its interior

   Note: $|(H_m \circ \cdots \circ H_1)(U)| \geq \alpha^m |U| > 2|U|$
4. $\beta := \min_\triangle \beta_\triangle$ ($\beta > 0$ because finitely many m-tuples $\triangle$)
5. $\boxed{\gamma} := \min(\frac{\varepsilon}{2}, \beta)$ ... one can show that γ is good.

3. Finite nonautonomous systems and non-shrinking of intervals

Example 2

All T_j are expanding ($\alpha_j > 1$). Theorem 1 does not work if $\alpha_j \geq 1$. To construct a counterexample, fix a small positive irrational ξ.



Some slopes are (in absolute value) > 1, but some are $= 1$.

$$(f_i)_{i=0}^{\infty} = \underbrace{\varphi, \varphi, \ldots, \varphi}_{k_1}, \psi_1, \underbrace{\varphi, \varphi, \ldots, \varphi}_{k_2}, \psi_2, \underbrace{\varphi, \varphi, \ldots, \varphi}_{k_3}, \psi_3, \ldots$$

where each ψ_i is either f or g. The sequence $k_1, k_2, k_3, \ldots$ and the maps $\psi_i \in \{f, g\}$ **can be chosen** in such a way that

$$\text{for } U = [0, \xi] \text{ we have } \lim_{n\to\infty} |f_0^n(U)| = 0.$$

4. Main result

Theorem 3

$T_1, T_2, \dots, T_n \colon [0,1] \to [0,1]$ *piecewise monotone, continuous, expanding, surjective,*

$B \subseteq \Sigma_n^+$ *a subshift which has the specification property and contains a periodic point* $\alpha = (\alpha_0\alpha_1 \dots \alpha_{p-1})^\infty$ *such that* $T_{\alpha_{p-1}} \circ \dots \circ T_{\alpha_1} \circ T_{\alpha_0}$ *is topologically mixing.*

Then

the step skew product on $B \times [0,1]$, $F(\omega, x) = (S(\omega), T_\omega(x))$, *has the specification property.*

Strategy of proof

1. Fix $\boxed{\varepsilon} > 0$ (we need $M = M(\varepsilon)$, gap length for F and ε)
2. Thm 1 $\Rightarrow \exists \gamma$ s.t. vertical int. $|U| \geq \varepsilon$ never shrinks below γ
3. $\widetilde{T} := T_{\alpha_{p-1}} \circ \dots \circ T_{\alpha_0}$ mixing, piec. monot. $\Rightarrow \exists m$ s.t. $\widetilde{T}^m(V) = [0,1]$ for every vertical int. $|V| \geq \gamma$
4. Spec. prop. in $B \Rightarrow \exists K = K(\varepsilon)$, gap length for $S|_B$ and ε
5. $\boxed{M} := mp + 2K$ ... can be shown to be gap length for F and ε

4. Main result

Corollary 4

$T_1, T_2, \ldots, T_n \colon [0,1] \to [0,1]$ *piecewise monotone, continuous, expanding,* ***mixing****,*

$B \subseteq \Sigma_n^+$ *a subshift which has a* ***fixed point****.*

Then

the corresponding step skew product has the specification property ***if and only if*** *the subshift has the specification property.*

Remark (Bertrand)

A subshift B has the specification property $\Longleftrightarrow$ it has a **uniform transition length**, *meaning that $\exists$ a positive integer M s.t.*

$$(\forall B\text{-words } u, v)(\exists B\text{-word } t \text{ of length } M)(utv \text{ is a } B\text{-word}).$$