Lyapunov non-typical behavior for cocycles through the lens of semigroup actions

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Motivation

- Birkhoff non-typical behavior
- Furstenberg & Oseledets theorems
- Finitely generated semigroup actions

Main results

- Statements
- Some remarks

Motivation

- Birkhoff non-typical behavior
- Furstenberg & Oseledets theorems
- Finitely generated semigroup actions

Birkhoff 'non-typical' behavior

(X, μ) probability space
$f : X \to X$ measure-preserving map
$\psi : X \to \mathbb{R}$ observable

THEOREM (Birkhoff 31') If $\psi \in L^1(\mu)$ then the limit

$$\tilde{\psi}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x))$$

exists for μ-almost every $x \in X$, and $\int \tilde{\psi}\, d\mu = \int \psi\, d\mu$.

DEF: A point $x \in X$ is 'non-typical' (or *irregular*) with respect to ψ if the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x))$ does not exist

DEF: The set of ψ-irregular points is denoted by $I_\psi(f)$

Birkhoff 'non-typical' behavior

Example

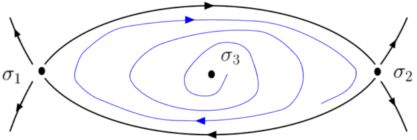


Figure: Irregular behavior on Bowen's eye

A DICHOTOMY: (Takens 94', 08', Barreira, Schmeling 00', Chen, Küpper, Shu 05', Li, Wu 13',...)

If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is $C^{1+\alpha}$-expanding map and $\psi : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is continuous then

$I_\psi(f) = \emptyset$ OR $I_\psi(f)$:
Full topological entropy
Full Hausdorff dim
Baire generic

RMK: Many works replacing *expansion* by *shadowing*, *specification*, or by weaker versions of these mechanisms...

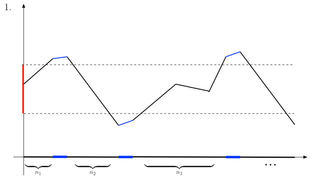
Birkhoff 'non-typical' behavior: some ideas

Existence of irregular points:

- find μ_1, μ_2 ergodic so that $\int \psi \, d\mu_1 \neq \int \psi \, d\mu_2$
- find x_i so that $\frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(x_i)) \to \int \psi \, d\mu_i$ $(i = 1, 2)$
- *uniform continuity* + *specification* $\Rightarrow$ there exists $z = \lim_n z_n$ where the orbit of z_n approximates well the finite orbits of x_1 and x_2 alternatively

$$\underbrace{00\cdots0}_{n_1}\underbrace{*\cdots*}_{\leqslant p}\underbrace{111\cdots1}_{n_2}\underbrace{*\cdots*}_{\leqslant p}\underbrace{0000\cdots0}_{n_3}\underbrace{*\cdots*}_{\leqslant p}\dots$$

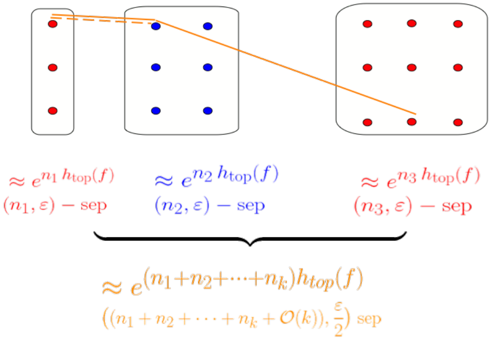
$n_1 \ll n_2 \ll n_3 \ll n_4 \ll \dots$ (arbitrary choice)



Birkhoff 'non-typical' behavior: some ideas

Existence of 'many' irregular points:

- if μ_1, μ_2 ergodic large entropy s.t. $\int \psi \, d\mu_1 \neq \int \psi \, d\mu_2$
- find many points x_i so that $\frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(x_i)) \to \int \psi \, d\mu_i$
- *uniform continuity* + *specification* $\Rightarrow$ there exist many irregular points $z = \lim_n z_n$ as before



- Similar reasoning yields a Baire generic set

Noncommuting random products

$G_1 = \{A_1, A_2, \ldots, A_\kappa\} \subset SL(d, \mathbb{R})$

$\mu = \nu^{\mathbb{Z}}$ Bernoulli probability measure on $\Sigma_\kappa := \{1, 2, \ldots, \kappa\}^{\mathbb{Z}}$

THEOREM (Furstenberg 63') Assume that:

1. the semigroup generated by G_1 is not contained in a compact subgroup of $SL(d, \mathbb{R})$,
2. the cocycle is strongly irreducible on suppν (ie, there is no finite family of proper subspaces of $\mathbb{R}^d$ invariant by A_i for every $i \in \operatorname{supp} \nu$).

Then $\lambda_+(A, \nu) > 0$

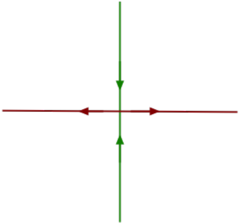
Top Lyapunov exponent, proved to exist by Furstenberg and Kesten 60':

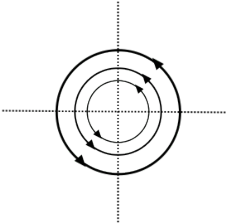
$$\lambda_+(A, \nu) = \lim_{n \to \infty} \frac{1}{n} \log \|A_{\omega_n} \ldots A_{\omega_2} A_{\omega_1}\| \quad \text{for } \mu - \text{a.e. } \omega$$

Rmks:

- Non-compactness and strong irreducibility are necessary conditions on the set of generating matrices

Example





$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda > 1 \qquad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

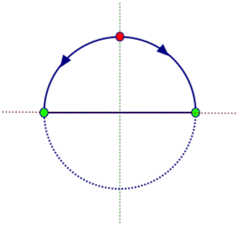
$$\mu = \left\{ \frac{1}{2}, \frac{1}{2} \right\}^{\mathbb{Z}}$$

Rmks:

- Non-compactness and strong irreducibility are necessary conditions on the set of generating matrices
- Projective action by bi-Lipschitz homeomorphisms on $X = \mathbf{P}\mathbb{R}^d$

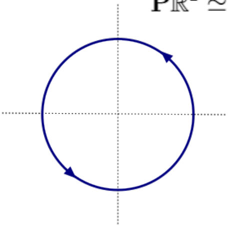
Example

$\mathbf{P}\mathbb{R}^2 \simeq \mathbb{S}^1$



$P_{A_1} : \mathbb{S}^1 \to \mathbb{S}^1$

Morse - Smale diffeomorphism



$P_{A_2} : \mathbb{S}^1 \to \mathbb{S}^1$

rotation by $\frac{\pi}{2}$

RMKS:

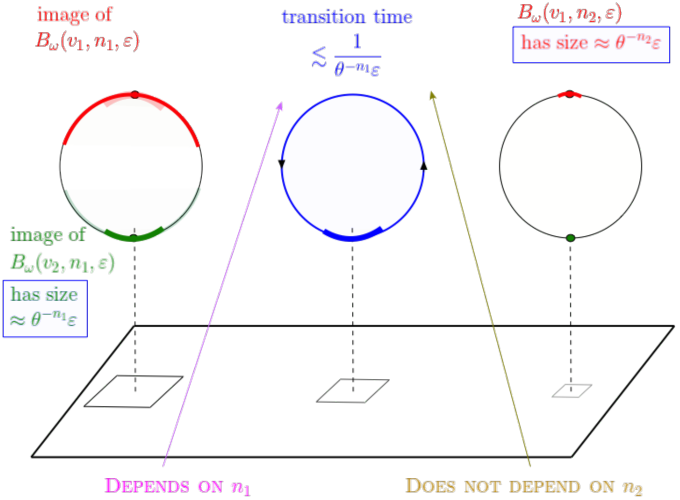
- Non-compactness and strong irreducibility are necessary conditions on the set of generating matrices
- Projective action by bi-Lipschitz homeomorphisms on $X = \mathbf{P}\mathbb{R}^d$
- Linear cocycles coded by skew-products

$$\begin{array}{rccccl} F_A\colon & \Sigma_\kappa \times \mathbb{R}^d & \longrightarrow & \Sigma_\kappa \times \mathbb{R}^d \\ & (x,v) & \mapsto & (f(x), A(x)\cdot v), \end{array}$$

$$\begin{array}{rccccl} P_A\colon & \Sigma_\kappa \times \mathbf{P}\mathbb{R}^d & \longrightarrow & \Sigma_\kappa \times \mathbf{P}\mathbb{R}^d \\ & (x,v) & \mapsto & \left(f(x), \frac{A(x)\cdot v}{\|A(x)\cdot v\|}\right), \end{array}$$

BEWARE: In general P_A does not satisfy the specification property (or any nonuniform version)
Sumi- V.-Yamamoto 16', Bomfim-Torres-V. 21'

Cartoon: Lack of specification for skew-products



ADVERTISEMENT: See Snoha's talk for positive results on the specification property for skew products of piecewise expanding

A multiplicative ergodic theorem

(X,μ) probability space, $f\colon X\to X$ invertible, measure-preserving

$A\colon X\to \mathrm{SL}(d,\mathbb{R})$ a measurable matrix-valued map

THEOREM (Oseledets 68') If $\log\|A^{\pm1}\|\in L^1(\mu)$ then for μ-a.e. x there exist $1\leqslant k(x)\leqslant d$ and:

- A-invariant splitting, measurable on x,
$$\Sigma_\kappa\times\mathbb{R}^d=E_x^1\oplus E_x^2\oplus\cdots\oplus E_x^{k(x)},$$
- real numbers (Lyapunov exponents)
$$\lambda_1(A,f,x)>\lambda_2(A,f,x)>\cdots>\lambda_{k(x)}(A,f,x)$$
s.t. if $A^{(n)}(x)=A(f^{n-1}(x))\cdots A(f(x))A(x)$ then

$$\lambda_i(A,f,x)=\lim_{n\to\pm\infty}\frac{1}{n}\log\|A^{(n)}(x)v_i\|,\quad \forall v_i\in E_x^i\setminus\{\vec{0}\}$$

RMKS: ◦ If μ is ergodic, the Lyapunov exponents are a.e. constant

◦ $\lambda_1(A,f,\mu)=\lambda_+(A,f,\mu)$

QUESTION: What can one say about the sets:

$$\left\{x \in X \colon \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \text{ does not exist}\right\}$$

(points in X that are irregular for the top Lyapunov exponent)

$$\left\{\mathbf{v} \in \mathbf{P}\mathbb{R}^d \colon \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\mathbf{v}\| \text{ does not exist, for some } x\right\}$$

(directions along which there exists irregular Lyapunov behavior)

$$\left\{v \in \mathbf{P}\mathbb{R}^d \colon \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\mathbf{x})v\| \text{ does not exist}\right\}$$

(Lyapunov irregular directions with respect to a fixed x)

$$\left\{x \in X \colon \left\{v \in \mathbf{P}\mathbb{R}^d \colon \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\mathbf{x})v\| \text{ does not exist}\right\} \text{ is 'large'}\right\}$$

(Points in X whose set of Lyapunov irregular directions is large)

Finitely generated semigroup actions

X compact metric space

$\psi : X \to \mathbb{R}$ continuous observable

$G_1 = \{id, f_1, f_2, \dots, f_\kappa\}$ bi-Lipschitz homeomorphisms on X

$G_n = \{g_{\omega_n} \circ \cdots \circ g_{\omega_2} \circ g_{\omega_1} : g_{\omega_j} \in G_1\}$
$G = \bigcup_{n \geqslant 1} G_n$ semigroup

QUESTION: What can one say about the convergence of the sequence

$$\frac{1}{n} \sum_{j=0}^{n-1} \psi(g_{\omega_j} \circ \cdots \circ g_{\omega_2} \circ g_{\omega_1}(x))$$

with respect to: (i) points in X, (ii) elements in $\omega \in \{1, 2, \dots, \kappa\}^{\mathbb{N}}$ which code the paths on the semigroup G

N.B. We do not consider pointwise ergodic theorems for the semigroup action (e.g. spherical averaging in free groups, ...)

ENTROPY:

X compact metric space

$G_1 = \{id, g_1, g_2, \ldots, g_\kappa\}$ continuous, $G = \bigcup_{n\geqslant 1} G_n$ semigroup

- $x, y \in X$ are (n,ε)-*separated* along the path $g_{\omega_n} \circ \cdots \circ g_{\omega_2} \circ g_{\omega_1}$ if there exists $1 \leqslant j \leqslant n$ s.t. $d(g_\omega^j(x), g_\omega^j(y)) > \varepsilon$
- Entropy of infinite path $\mathcal{F}_\omega = (g_\omega^j)_j$ in G (Kolyada-Snoha 96'):

$$h(\mathcal{F}_\omega) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(\omega, n, \varepsilon)$$

  where $s(\omega, n, \varepsilon) =$ max. card. of (n,ε)-separated points along path
- GLW-entropy of semigroup action (Ghys-Langevin-Walczak 88'):

$$h^{GLW}(\mathbb{S}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(G, n, \varepsilon)$$

  where $s(G, n, \varepsilon) =$ max. card. of points separated by G_n elements
- B-entropy of free semigroup action (Bufetov 99'):

$$h^{B}(\mathbb{S}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \Big(\frac{1}{\kappa^n} \sum_{g\in G_n} s(\omega, n, \varepsilon) \Big)$$

Main results

(joint with G. Ferreira - UFMA, Brazil)

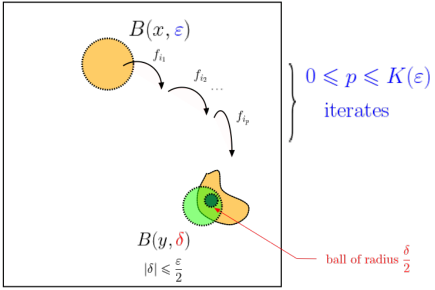
Linear cocicles & Semigroup actions

$G_1 = \{id, f_1, f_2, \ldots, f_\kappa\}$ continuous maps on compact metric X

$G_n = \{g_{\omega_n} \circ \cdots \circ g_{\omega_2} \circ g_{\omega_1} : g_{\omega_j} \in G_1\}$

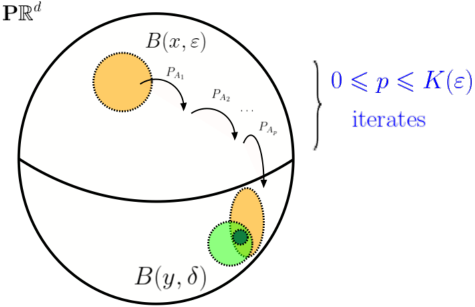
$G = \bigcup_{n \geqslant 1} G_n$ semigroup

DEF: The semigroup action generated by G_1 has frequent hitting times if for any $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ so that for any balls (B_1, B_2) with $|B_1| = \varepsilon$ and $0 < |B_2| \leqslant \frac{\varepsilon}{2}$ there exists $0 \leqslant p \leqslant K(\varepsilon)$, an element $\underline{\omega} \in \Sigma_\kappa := \{1, 2, \ldots, \kappa\}^{\mathbb{N}}$ and a ball $B_2' \subset B_2$ of radius $|B_2|/2$ so that $f_{\underline{\omega}}^p(B_1) \supset B_2'$.



$G_1 = \{A_1, A_2, \ldots, A_\kappa\} \subset SL(d, \mathbb{R})$

DEF: The semigroup generated by G_1 is strongly projectively accessible the semigroup generated by the projective linear maps satisfies the frequent hitting times condition.



LEMMA: *If $G_1 = \{f_1, f_2, \dots, f_\kappa\}$ admits a subset $\hat{G}_1$ which acts minimaly by isometries then the semigroup action generated by G_1 satisfies the frequent hitting times property.*

Examples

The semigroup action generated by G_1 has frequent hitting times:

- $G_1 = \{f_1, f_2, \dots, f_\kappa\}$ homeomorphisms on $\mathbb{T}^d$ $(d \geqslant 1)$ where some $f_j : \mathbb{T}^d \to \mathbb{T}^d$ is a irrational translation
- $G_1 = \{f_1, f_2, \dots, f_\kappa\}$ homeomorphisms on $\mathbb{T}^d$ $(\kappa \geqslant d \geqslant 1)$ where $f_{j_i}(x) = x + v_i$, $v_i \notin \mathbb{Q}^d$ and $\{v_1, \dots, v_d\}$ base in $\mathbb{R}^d$
- $G_1 = \{f_1, f_2\}$ where $f_i : \mathbf{P}\mathbb{R}^3 \to \mathbf{P}\mathbb{R}^3$ are projective maps $(\alpha, \beta \notin \mathbb{Q})$

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \in SO(3, \mathbb{R})$$

- $G_1 = \{f_1\}$, $f_1 : \mathbf{P}\mathbb{R}^4 \to \mathbf{P}\mathbb{R}^4$ projective map (Baire generic & full Lebesgue on $\{(a, b, c, d) \colon a^2 + b^2 = 1, \ c^2 + d^2 = 1\}$)

$$\begin{pmatrix} a & 0 & -b & 0 \\ 0 & c & 0 & -d \\ b & 0 & a & 0 \\ 0 & d & 0 & c \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{R})$$

Linear cocycles

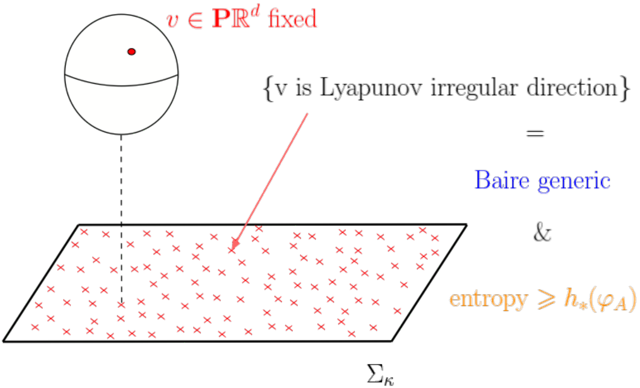
THEOREM 1 Assume that the semigroup generated by $G_1 = \{A_1, A_2, \dots, A_\kappa\} \subset SL(d, \mathbb{R})$

1. is not contained in a compact subgroup of $SL(d, \mathbb{R})$,
2. is strongly projectively accessible

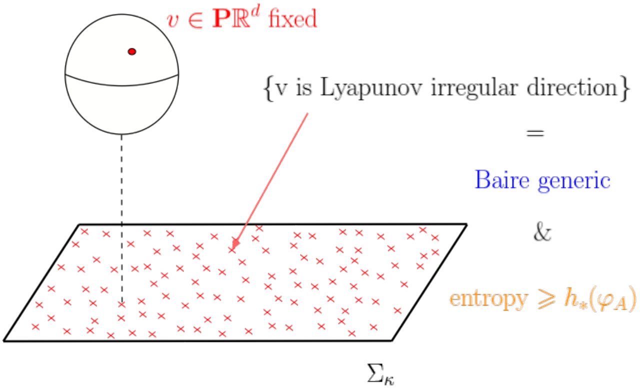
Then for each $v \in \mathbf{P}\mathbb{R}^d$ there exists $\mathcal{R}_v \subset \Sigma_\kappa$ Baire generic, with entropy at least $h_*(\varphi_A)$ s.t. for every $\omega \in \mathcal{R}_v$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| < \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| \qquad (\star)$$

Cartoon version:



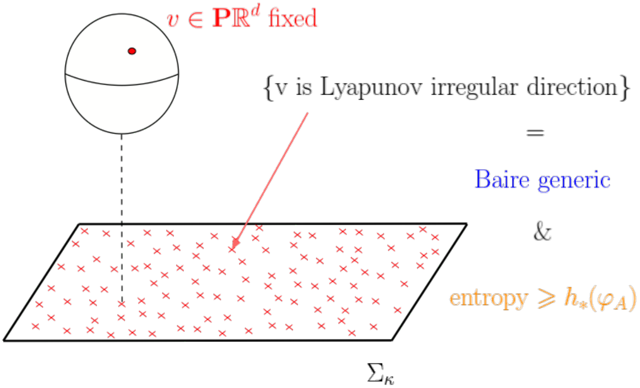
Cartoon version:



$\varphi_A : \Sigma_\kappa \times \mathbf{P}\mathbb{R}^d \to \mathbb{R}$ given by $\varphi_A(\omega, v) = \log \frac{\|A(\omega)v\|}{\|v\|}$

$$h_*(\varphi_A) = \sup \Big\{ c \geqslant 0 \colon \text{there exist } \mu_1, \mu_2 \in \mathcal{M}_{erg}(P_A) \text{ so that } h_{\pi_* \mu_i}(\sigma) \geqslant c \text{ and } \int \varphi_A \, d\mu_1 < \int \varphi_A \, d\mu_2 \Big\}$$

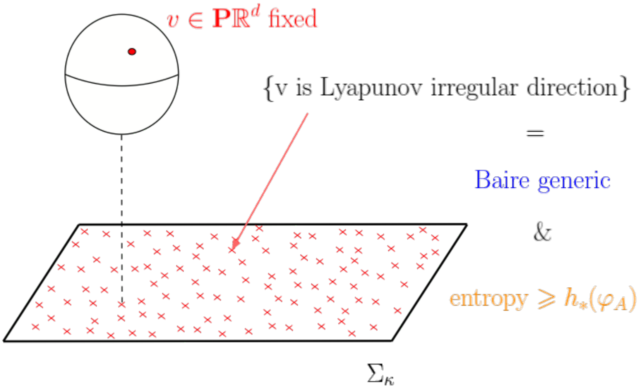
THEOREM 1:



RMK:

$$h_*(\varphi_A) \geqslant \sup \left\{ h_\nu(\sigma) \colon \nu \in \mathcal{M}_{erg}(\sigma) \text{ and } \lambda_+(A, \nu) > \inf_{\eta \in \mathcal{M}_{erg}(\sigma)} \lambda_+(A, \eta) \right\}$$

Theorem 1:



Corollary 1: Under the previous assumptions, there exists a Baire residual subset $\mathcal{R} \subset \Sigma_\kappa$ and a dense subset $\mathcal{D} \subset \mathbf{P}\mathbb{R}^d$ so that $(\star)$ holds for every $\omega \in \mathcal{R}$ and every $v \in \mathcal{D}$.

THEOREM 2: Let $A : \Sigma_\kappa \to SL(3,\mathbb{R})$ be a locally constant and hyperbolic cocycle, and $\Sigma_\kappa \times \mathbb{R}^3 = E^s \oplus E^u$ satisfy $\dim E^s = 2$. There is a C^0-open neighborhood $\mathcal{U} \subset C_{\mathrm{loc}}(\Sigma_\kappa, SL(3,\mathbb{R}))$ of A and C^0-open sets $\mathcal{U}_1 \cup \mathcal{U}_2$ dense in $\mathcal{U}$ so that

1. every $B \in \mathcal{U}_1$ admits a continuous splitting $\mathbb{R}^3 = E^u_{B,\omega} \oplus E^{s,1}_{B,\omega} \oplus E^{s,2}_{B,\omega}$, and there exists a C^0*-open and dense, full Haar measure subset* $\mathcal{O} \subset \mathcal{U}_1$ s.t. if $B \in \mathcal{O}$ then the set of Lyapunov irregular points in Σ_κ is Baire generic and has full entropy
2. there exists a C^0*-Baire residual and full Haar measure subset* $\mathcal{R} \subset \mathcal{U}_2$ s.t. if $B \in \mathcal{R}$ then there exists a Baire residual $S \subset \Sigma_\kappa$ and for each $\omega \in \mathcal{S}$ there is a dense $\mathcal{D}_\omega \subset E^s_\omega$ s.t. for every $v \in \mathcal{D}_\omega$
$$\liminf_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)v\| < \limsup_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)v\|.$$

Related work:

- Previous results on irregular behavior for the *top Lyapunov exponent* of Hölder continuous cocycles by Herman 81', Furman 97' for cocycles over minimal homeomorphisms, by Tian 15', 17' for linear cocycles over mixing hyperbolic basic set for $C^{1+\alpha}$ diffeomorphism, and by Carvalho, V. 19' on Baire genericity of Birkhoff averages and top Lyapunov exponents

  $\longrightarrow$ informations on the shift space

- Díaz, Gelfert, Rams 19' use a notion of transitions on $SL(2,\mathbb{R})$ cocycles with the 'flavor' of frequent hitting times to study multifractal analysis of step $SL(2,R)$ cocycles

- Previous results on Hölder cocycles (i) explore u.s.c. of $\mu \mapsto \lambda_+(A,f,\mu)$, (ii) bounded distortion results for linear cocycles by Kalinin 11'

Semigroup actions

Theorem 3 Let X be a compact metric space and $\psi : X \to \mathbb{R}$ be a continuous observable. Assume that:

1. semigroup action by the bi-Lipschitz homeos $G_1 = \{f_1, f_2, \dots, f_\kappa\}$ has frequent hitting times
2. ψ is *not a coboundary w.r.t. some* f_i

Then:

$$I_\psi(\mathbb{S}) := \left\{ x \in X : \frac{1}{n}\sum_{j=0}^{n-1} \psi(g_\omega^j(x)) \text{ diverges along some path in G} \right\}$$

is Baire generic in X.

Theorem 4 Let X be a compact metric space and $\psi : X \to \mathbb{R}$ be a continuous observable. Assume that:

1. semigroup action by the bi-Lipschitz homeos $G_1 = \{f_1, f_2, \ldots, f_\kappa\}$ has frequent hitting times
2. $\varphi_\psi(\omega, x) := \psi(x)$ is *not a coboundary w.r.t F*

Then

$$I_\psi(\mathbb{S}) := \left\{ x \in X : \frac{1}{n}\sum_{j=0}^{n-1} \psi(g_\omega^j(x)) \text{ diverges along some path in G} \right\}$$

satisfies:

$h^{GLW}(\mathbb{S}, I_{\mathbb{S}}(\psi)) \geqslant H^{\text{Pinsker}}(\psi)$ and $h^B(\mathbb{S}, I_{\mathbb{S}}(\psi)) \geqslant h_*(\varphi_\psi)$

$H^{\text{Pinsker}}(\psi) = c$ iff for every $\varepsilon > 0$ there exist $\mu_1, \mu_2 \in \mathcal{M}_{erg}(F)$ which distinguish ψ and $h_{\mu_i}(F \mid \sigma) > c - \varepsilon$

$h^{GLW}(\mathbb{S}, \cdot)$ = entropy 'coherent' to Ghys, Langevin, Walczak 88', Biś 04'

$h^B(\mathbb{S}, \cdot)$ = entropy 'coherent' to Bufetov 99'

THEOREM 5 Let X be a compact metric space and $\psi : X \to \mathbb{R}$ be a continuous observable. Assume that:

1. semigroup action by the bi-Lipschitz homeos $G_1 = \{f_1, f_2, \dots, f_\kappa\}$ has frequent hitting times
2. $\varphi_\psi(\omega, x) := \psi(x)$ is not a coboundary w.r.t F
3. there exists $1 \leqslant i \leqslant \kappa$ so that f_i is *minimal*

Then:

- $h_I^{fiber}(\mathbb{S}, \psi) := \sup_{\omega \in \Sigma_\kappa} h_{I_\omega(\psi)}(\mathcal{F}_\omega) \geqslant H^{\text{Pinsker}}(\psi)$ and
- $\{\omega \in \Sigma_\kappa : h_{I_\omega(\psi)}(\mathcal{F}_\omega) \geqslant H^{\text{Pinsker}}(\psi)\}$ has entropy larger or equal than $H_\sigma^{\text{Pinsker}}(\psi)$

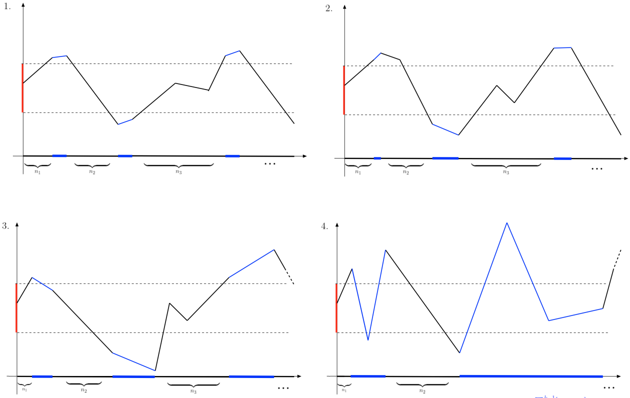
$H^\sigma(\psi) = c$ iff for every $\varepsilon > 0$ there exist $\mu_1, \mu_2 \in \mathcal{M}_{erg}(F)$ which distinguish ψ and $h_{\pi_* \mu_i}(\sigma) > c - \varepsilon$

Few remarks

- variable transition times as in the gluing orbit property (or very weak specification)
- transition times much larger than all the orbit past
- synchronization between fiber and base dynamics in skew-products
- construction of shadowing orbits much different than specification/shadowing
- fibered entropy distribution principle
- relations between entropy of semigroups and skew-products

Thank you!

Děkuju   Köszönöm   Dziękuję   Ďakujem



$\lim_{k\to\infty} \frac{p_k}{n_k} = 0$

$\lim_{k\to\infty} \frac{\sum_{j=1}^{k-1}[n_j + p_j]}{n_{k+1}} = 0$