Complexity of some orbit equivalence relations induced by homeomorphism groups

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Prague, June 14-18, 2021

Introduction

The aim of the talk:

1) to present a formal framework for classification results

   Descriptive Set Theory

   - Polish spaces / standard Borel spaces
   - continuous / Borel mappings

   Invariant Descriptive Set Theory:

   - equivalence relations on standard Borel spaces
   - Borel reductions

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   - Borel reductions

2) apply to objects in Topological Dynamics and Continuum Theory.

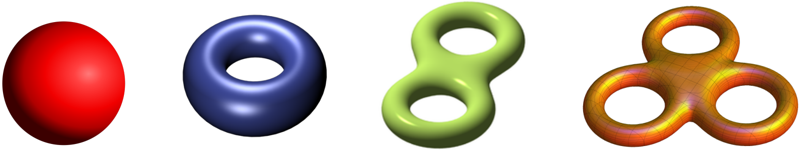
Theorem

Interval dynamical systems (up to conjugacy) can be classified by countable structures (up to isomorphism).

Example of a classification result

Theorem

2-dimensional compact orientable manifolds without boundary are up to homeomorphism classified by non-negative integers.



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Example of a classification result

Theorem

2-dimensional compact orientable manifolds without boundary are up to homeomorphism classified by non-negative integers.

There exists a natural mapping

$$f : X = \{\text{2-dimensional compact orientable manifolds}\} \to Y = \{0, 1, \ldots\}$$

such that

- homeomorphic manifolds are mapped to equal numbers,
- non-homeomorphic manifolds are mapped to distinct numbers.

Complexity: reductions

Definition

- Let X, Y be sets,
- let E, F be equivalence relations (ER) on X and Y respectively.

  A mapping $f\colon X \to Y$ is called a reduction from E to F if for every two points $x, x' \in X$ we have $xEx' \iff f(x)Ff(x')$.

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Definition (natural reducibility)

- Let X, Y be Polish (standard Borel spaces),
- let E, F be ERs on X, Y respectively.

We say that E is Borel reducible to F, ($E \leq_B F$), if there is a Borel measurable reduction from E to F.

We say that E is Borel bireducible with F, ($E \sim_B F$), if $E \leq_B F$ and $F \leq_B E$.

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Theorem (essentially: Banach, Stone)

Homeomorphism ER on compact metrizable spaces is Borel reducible into isometry ER of separable Banach spaces.

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- X = collection of all metrizable compacta
- Y = collection of all separable Banach spaces
- E = homeo ER on X
- F = isometry ER on Y

$$\begin{aligned} f\colon\ & X \to Y, \\ & K \mapsto C(K, \mathbb{R}) \end{aligned}$$

- K homeo L iff $C(K, \mathbb{R})$ isometric to $C(L, \mathbb{R})$.

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- f Borel?!

Coding

- a class of structures usually forms a proper class
- coding = a set of representatives with a Polish topology (standard Borel structure) on this set
- necessary so that we can verify borelness of a reduction
- countable groups can be coded by functions $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

  hence elements of $\mathbb{N}^{\mathbb{N} \times \mathbb{N}}$
- the hyperspace of the Hilbert cube $[0, 1]^{\mathbb{N}}$

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- usually, there are more natural coding for essentially the same thing
- all known natural coding of the same thing turned out to be equivalent (with respect to $\sim_B$)

Orbit ERs

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Theorem (Becker, Kechris)

For every Polish group G there is a G-action on a Polish space X such that $E_G^Y \leq_B E_G^X$ for every G-action on a Polish space Y.

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- denote by E_G any such E_G^X (unique up to $\sim_B$)
- (Uspenskii) there is a universal Polish group G_∞.
- Every orbit ER can be Borel reduced to E_{G_∞} (this follows by Hjorth-Mackey: extensions of group actions)
- We consider these groups $G_\infty, S_\infty, F_2(F_\infty)$

Complexity degrees and benchmarks

- universal orbit ER: E_{G_∞}
  - the most complex among all orbit ERs
  - isometry ER of complete separable metric spaces (Gao, Kechris)
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- equality of countable sets: $E_{=^+}$
  - the most complex S_∞-orbit ER which is Π^0_3
  - $(x_n), (y_n) \in \mathbb{R}^\mathbb{N}$ are equivalent iff $\{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}$
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  - isomorphism ER of countable locally finite acyclic graphs
- universal countable ER $=$ F_2-universal orbit ER: E_{F_2}
  - the most complex Borel ER with countable orbits (Feldman-Moore)
  - isometry ER of Heine-Borel complete metric spaces

$$E_{F_2} \lneq_B E_{=^+} \lneq_B E_{S_\infty} \lneq_B E_{G_\infty}$$

Homeomorphism ER and conjugacy ER

- X ... compact metrizable space
- $C(X)$ ... continuous selfmaps of X into itself
- $\mathcal{K}(X)$ ... hyperspace of compact subsets of X
- $\mathcal{H}(X)$ ... homeomorphisms of X onto itself

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a) Conjugacy action

$$\mathcal{H}(X) \curvearrowright C(X), \quad h \cdot f = h^{-1} \circ f \circ h$$

b) 'Shift' (ambient homeomorphism) action

$$\mathcal{H}(X) \curvearrowright \mathcal{K}(X), \quad h \cdot K = h(K)$$

Classification results up to conjugacy

Theorem (Hjorth, 2000)

- *Conjugacy ER of invertible interval DS* $\sim_B E_{S_\infty}$.
- *Conjugacy ER of invertible DS on* $[0,1]^2$ *is NOT* $\leq_B E_{S_\infty}$.

Classification results up to conjugacy

Theorem

2001 Camerlo-Gao: Conjugacy ER of (invertible) Cantor set DS $\sim_B E_{S_\infty}$

2009 Clemens: Conjugacy of subshifts $\sim_B E_{F_2}$

2017 Kaya: Conjugacy ER of pointed invertible minimal Cantor DS $\sim_B E_{=^+}$

2017 Kaya: $E_{=^+} \leq_B$ *conjugacy of invertible minimal Cantor DS*

2019 Kwietniak: Conjugacy ER of shifts with specification is $\sim_B E_{F_2}$

Classification results up to conjugacy

Theorem (Bruin, V., 2019)

Conjugacy ER of interval DS $\sim_B E_{S_\infty}$.

Proof.

- $\geq_B$: Hjorth (even for homeos)
- $\leq_B$: tent map example

□

Conjugacy ER on $C([0,1]) \leq_B E_{S_\infty}$ - proof

- Left strict local maximum of f.
- M_f ... all left/right strict local max/min of f.
- C_f ... the smallest set such that
  a) $\{0,1\} \cup M_f \subseteq C_f$,
  b) if $f^{-1}(y)$ contains an interval then $y \in C_f$,
  c) if $n \in \mathbb{N}$ then $\mathrm{Acc}(\mathrm{Fix}(f^n)) \subseteq C_f$,
  d) $f(C_f) \subseteq C_f$,
  e) if $y \in C_f$ then $\mathrm{Acc}(f^{-1}(y)) \subseteq C_f$.

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  d) $f(C_f) \subseteq C_f$,
  e) if $y \in C_f$ then $\mathrm{Acc}(f^{-1}(y)) \subseteq C_f$.
- C_f is countable, it is determined only by the dynamics of f.
- The closure of C_f captures interesting dynamics of f.
- D_f ... a countable invariant dense subset of $[0,1] \setminus \mathrm{Cl}(C_f)$.
- $\varphi\colon f \mapsto (C_f \cup D_f, <\restriction_{C_f \cup D_f}, f \restriction_{C_f \cup D_f})$ is a reduction.
- Isomorphism ER of countable structures $\leq_B E_{S_\infty}$.

Hjorth's conjecture

Every ER which is induced by a continuous action of the group $\mathcal{H}([0,1])$ is Borel reducible to E_{S_∞}, i.e.

$$E_{\mathcal{H}([0,1])} \sim_B E_{S_\infty}.$$

Classification results up to conjugacy

Theorem (Bruin, V., 2019)

Conjugacy ER of Hilbert cube DS $\sim_B E_{G_\infty}$.

Proof.

- $\leq_B$: by Rosendal, Zielinski 2018
- $\geq_B$: Lemma: Let K be compact. Then there is a compactification of $\mathbb{N}$ with remainder K. Every homeo of K can be extended to a homeo of the whole compactification.

□

Homeomorphism ER

'Shift' action $\mathcal{H}(X) \curvearrowright \mathcal{K}(X), \quad h \cdot K = h(K)$
The corresponding orbit equivalence relation is ambient homeomorphism.

Theorem (Kechris, Solecki)

Homeo ER on compacta $\leq_B$ orbit ER.

Proof.

- $Q := [0,1]^{\mathbb{N}}$
- $Z := \{x \in Q : x_1 = 0\}$.
- Z is so called Z-set in Q, thus for every compact $K, L \subseteq Z$, every homeo $h : K \to L$, can be extended to a homeo of Q
- $id : \mathcal{K}(Z) \to \mathcal{K}(Q)$ is the desired reduction.

□

Infinite-dimensional compacta

Definition

- continuum = compact and connected space
- locally connected (LC) = each point has small connected neighborhoods
- absolute retract (AR) = retract of the Hilbert cube $[0,1]^{\mathbb{N}}$

AR $\implies$ LC continuum $\implies$ continuum $\implies$ compactum

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Theorem

2016 Zielinski: Homeo ER on compacta $\sim_B E_{G_\infty}$

2017 Chang, Gao: Homeo ER on continua $\sim_B E_{G_\infty}$

2018 Cieśla: Homeo ER on LC *continua* $\sim_B E_{G_\infty}$

2018 Krupski, V.: Homeo ER on AR $\sim_B E_{G_\infty}$

Some methods

Theorem (Krupski, V.)

Homeo ER on compacta $\leq_B$ homeo ER on AR.

Proof.

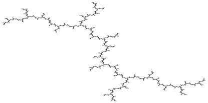
- Lemma: Let K be compact. Then there is a compactification of $\mathbb{N}$ with remainder K. Every homeo of K can be extended to a homeo of the whole compactification.
- $f : \mathcal{K}(Q) \to \mathcal{K}(Q \times [0,1] \times [0,1])$

□

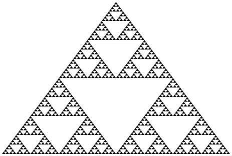
Low-dimensional topological classes

Definition

- 0-dimensional compacta
  = closed subspaces of the Cantor set
- dendrites
  = 1-dimensional AR
- rim-finite (RF) continua
  = have a base with finite boundaries
- rim-finite compacta
- $\mathrm{AR}(\mathbb{R}^2)$ = embeddable in the plane

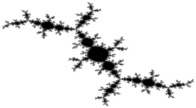


a dendrite



a rim-finite continuum

$$\text{Dendrite} \iff \text{RF and } \mathrm{AR}(\mathbb{R}^2)$$
$$\text{0-dimensional} \implies \text{RF}$$



$\mathrm{AR}(\mathbb{R}^2)$

Classification results up to homeomorphism

We are omitting: 'Homeomorphism ER on'

Theorem

2001 Camerlo, Gao: 0-dimensional compacta $\sim_B E_{S_\infty}$ *(Stone duality)*

2005 Camerlo, Darji, Marcone: dendrites $\sim_B E_{S_\infty}$ *(ternary relations)*

2018 Krupski, V.: rim-finite continua $\sim_B E_{S_\infty}$ *(n-ary relations)*

2019 Dudák, V.: $\mathrm{AR}(\mathbb{R}^2) \sim_B E_{S_\infty}$ *(reduction to the boundary)*

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2019 Dudák, V.: $\mathrm{AR}(\mathbb{R}^2) \sim_B E_{S_\infty}$ *(reduction to the boundary)*

2019 Dudák, V.: $\mathrm{AR}(\mathbb{R}^3)$, $\mathrm{LC}(\mathbb{R}^2) \not\sim_B E_{S_\infty}$

2018 Krupski, V.: rim-finite compacta $\not\sim_B E_{S_\infty}$ *(Hjorth: turbulence)*

rim-finite cases:

Non-commutative topology: C^*-algebras excursion

Gelfand duality: Compact spaces up to homeo = separable unital commutative C^*-algebras up to isomorphism

Theorem (Sabok)

Isomorphism ER of separable C^*-algebras is Borel bireducible to E_{G_∞}.

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Gelfand duality: Compact spaces up to homeo = separable unital commutative C^*-algebras up to isomorphism

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Definition

A C^*-algebra $P \in \mathcal{M}$ is *projective in class* $\mathcal{M}$ if for every $B \in \mathcal{M}$, every quotient morphism $\pi \colon B \to B/J$ and $f \colon P \to B/J$ there is a morphism $\bar{f} \colon P \to B$ such that $f = \pi \bar{f}$.

- X is AR $\iff$ $C(X, \mathbb{C})$ is projective in the class of commutative unital C^*-algebras (follows simply by the Gelfand duality)
- X is a dendrite $\iff$ $C(X, \mathbb{C})$ is projective in the class of unital C^*-algebras (Chigogidze, Dranishnikov 2010)

Questions

What is the exact complexity of conjugacy ER of

- triangular maps?
- transitive interval DS?
- minimal Cantor DS? . . .

THANK YOU FOR YOUR ATTENTION.