Geometry and Computability of zero-temperature measures

Christian Wolf
The City College of New York

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Outline of the talk

1. Pressure, equilibrium states and zero-temperature measures
2. Existence of zero-temperature measures
3. Computability of zero-temperature measures and residual entropy

Geometry and Computability of zero-temperature measures

Setup: Let (X, d) be a compact metric space and let $f : X \to X$ be continuous. We endow

$$\mathcal{M} = \{\mu : f - \text{invariant Borel probability measure on } X\}$$

with the weak$*$ topology.

$\implies$ $\mathcal{M}$ a compact, convex, metrizable.

Let $\mathcal{M}_E \subset \mathcal{M}$ be the subset of ergodic measures, that is, if $f^{-1}(A) = A$ then $\mu(A) \in \{0, 1\}$.

We denote by $h_\mu(f)$ the measure-theoretic entropy of μ.

Let $C(X,\mathbb{R})$ denote the space of continuous functions $\phi : X \to \mathbb{R}$ endowed with the supremums norm.

Definition

The *topological pressure* $P_{\mathrm{top}} : C(X,\mathbb{R}) \to \mathbb{R}$ is defined by

$$P_{\mathrm{top}}(\phi) = \sup_{\mu \in \mathcal{M}} h_\mu(f) + \int \phi \, d\mu$$

A measure μ is an *equilibrium state* of ϕ if $P_{\mathrm{top}}(\phi) = h_\mu(f) + \int \phi \, d\mu$. The topological pressure of the constant zero potential is called the topological entropy $h_{\mathrm{top}}(f)$ of f.

We assume that the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous which guarantees that for each potential ϕ there exists at least one (ergodic) equilibrium state.

Let $\phi \in C(X, \mathbb{R})$ such that for all $\beta \geq 0$ the potential $\beta\phi$ has a unique equilibrium state $\mu_\beta = \mu_{\beta\phi}$. We think of $\beta = \frac{1}{T}$ as the inverse temperature (T being the temperature) of the system.

Definition

If the limit $\mu_{\infty,\phi} = \lim_{\beta\to\infty} \mu_\beta$ (in the weak$*$ topology) exists we call $\mu_{\infty,\phi}$ the zero-temperature measure of ϕ. If $\mu_{\infty,\phi}$ does not exist we consider the set of of accumulation points of μ_β as $\beta \to \infty$ which we call the ground state set $GS(\phi)$ of ϕ.

Theorem (Folklore)

Suppose $\mu \in GS(\phi)$. *Then*

(i) $\int \phi \, d\mu = \sup\{\int \phi \, d\nu : \nu \in \mathcal{M}\} \stackrel{\text{def}}{=} b_\phi$.

(ii) $h_\mu(f) = \sup\{h_\nu(f) : \int \phi \, d\nu = b_\phi\} \stackrel{\text{def}}{=} h_{\infty,\phi}$.

The quantity $h_{\infty,\phi}$ is called the residual entropy of ϕ, i.e., the maximal entropy at zero temperature.

Let $d \geq 2$, $\mathcal{A} = \{0, \dots, d-1\}$ be a finite alphabet and let A be a transition matrix. We define $X = X_A = \{(x_i)_{i \in \mathbb{N}_0} : x_i \in \mathcal{A}, A_{x_i, x_{i+1}} = 1\}$. Let $f : X \to X$ be the shift map. We say f is a subshift of finite type. In the following we assume that f is transitive.

Denote by $LC(X, \mathbb{R}) = \bigcup_k LC_k(X, \mathbb{R})$ the locally constant potentials, where $LC_k(X, \mathbb{R})$ are the potentials constant on cylinders of length k.

Theorem (Brémont, Nonlinearity, 2003)

If $\phi \in LC(X, \mathbb{R})$ then the zero-temperature measure $\mu_{\infty,\phi}$ exists.

Theorem (W., Yang, TAMS, 2019)

Let $k \in \mathbb{N}$. Then there exist a partition of $LC_k(X, \mathbb{R})$ into finitely many convex cones $\mathcal{U}_1, \dots, \mathcal{U}_{\ell_1}, \mathcal{U}_{\ell_1+1}, \dots, \mathcal{U}_N$ and k-elementary points $x_1, \dots, x_\ell$ such that:

(i) *$\mathcal{U}_1, \dots, \mathcal{U}_{\ell_1}$ are open, and $\overline{\mathcal{U}_1 \cup \dots \cup \mathcal{U}_\ell} = LC_k(X, \mathbb{R})$. Further, the orbit of x_i is the unique maximizing periodic orbit of ϕ. For each $i = 1, \dots, \ell$, for all $\phi \in \mathcal{U}_i$, we have $\mu_{\infty,\phi} = \mu_{x_i}$, i.e., the unique invariant measure supported on the periodic orbit x_i.*

(ii) *If $\phi \in \mathcal{U}_{\ell+1} \cup \dots \cup \mathcal{U}_N$ then $\mu_{\infty,\phi}$ is either non-ergodic or has positive entropy or both.*

Basics from Computable Analysis:

Definition

Let $m \in \mathbb{N}$ and $x \in \mathbb{R}^m$. An *oracle* of x is a function $\psi : \mathbb{N} \to \mathbb{Q}^m$ such that $\|\psi(n) - x\| < 2^{-n}$. Moreover, x is computable if there is a Turing Machine (a computer program for our purposes) ψ which is an oracle of x.

Basic Facts:

(i) Rational numbers, algebraic numbers, and some transcendental numbers such as e and π are computable real numbers.

(ii) There are only countably many computable points in $\mathbb{R}^m$.

Definition

Let $S \subset \mathbb{R}^m$. A function $g : S \to \mathbb{R}$ is *computable* if there is a Turing machine χ so that for any $x \in S$ and any oracle ψ for x, $\chi(\psi, n)$ is a rational number so that $|\chi(\psi, n) - g(x)| < 2^{-n}$.

Definition

Let $S \subset \mathbb{R}^m$. A function $g : S \to \mathbb{R}$ is *upper semi-computable* if there is a Turing machine χ so that for any $n \in \mathbb{N}$ and $x \in S$, and any oracle ψ for x, $\chi(\psi, n) = q_n \in \mathbb{Q}$ such that $q_n \downarrow g(x)$ as $n \to \infty$. Analogously one defines lower semi-computable functions.

Lemma

A function $g : S \to \mathbb{R}$ is computable if and only if g is lower and upper semi-computable.

Similar notions of computability exist for many mathematical objects (subsets of Euclidian space, shift spaces, measure spaces, etc.) and functions.

A Computability result for zero-temperature measures:

Theorem (Burr, W., Nonlinearity 2021)

Let $k \in \mathbb{N}$. Let $\mathcal{O}_k$ be the set of potentials $\phi \in LC_k(X, \mathbb{R})$ such that the zero-temperature measure $\mu_{\infty,\phi}$ is supported on a periodic orbit. Then

- (i) *Then $\mathcal{O}_k \ni \phi \mapsto \mu_{\infty,\phi}$ is computable.*
- (ii) *The set $\mathcal{O}_k$ is a lower computable open and dense subset of $LC_k(X, \mathbb{R})$.*
- (iii) *If $\phi_0 \in LC_k(X, \mathbb{R}) \setminus \mathcal{O}_k$, then μ_{∞,ϕ_0} is not computable.*
- (iv) *If the cylinder length k is not given in advance, then μ_{∞,ϕ_0} is not computable for any $\phi_0 \in LC(X, \mathbb{R})$.*

A Computability result for the residual entropy:

Theorem (Burr, W., Nonlinearity 2021)

Let $f : X \to X$ be a transitive subshift of finite type. Then the function $\phi \mapsto h_{\infty,\phi}$ is upper semi-computable, but not computable on $C(X, \mathbb{R})$. Moreover, the map $\phi \mapsto h_{\infty,\phi}$ is continuous at ϕ_0 if and only if $h_{\infty,\phi_0} = 0$.